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THEORETICAL STUDIES

ON THE

FLOW THROUGH NOZZLES

AND RELATED PROBLEMS

WITH THE AP[illegible] OF THE CHAIRMA[illegible] COMMITTEE [illegible] AND DEVELOPMENT.

A Report Submitted

by the

Applied Mathematics Group, New York University

to the

Applied Mathematics Panel

National Defense Research Committee

April 1944

[redacted]

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THEORETICAL STUDIES ON THE FLOW THROUGH NOZZLES AND RELATED PROBLEMS

(by R. Courant and K.O. Friedrichs)

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SUMMARY

The present report summarizes the results of a study which the New York University Group of the Applied Mathematics Panel has undertaken upon the request of the Bureau of Aeronautics, Navy Department (Project No. NA-167), and which was carried out under the responsibility of Professor K.O.Friedrichs with the assistance of Dr. Chas. R. DePrima and other members of the group. Our group is greatly indebted to Mr. E. S. Roberts of the American Cyanamid Company for his stimulating advice.

The original request was for assistance in the design of unconventional nozzles for rocket motors, with a view toward attaining shortness while preserving high efficiency. Since the opening angle of such nozzles is necessarily wide and the expansion rapid, it was necessary to improve the classical (hydraulic) theory of the de Laval nozzle and to widen the scope of the investigation further by consideration of gas dynamical phenomena more generally. Therefore, the present report is not concerned solely with the original problem of exhaust nozzles, but it contains methods and results of potential interest for other problems; in particular, attention is given to the design of "kinetic compressors", i.e., of nozzles serving, not as means

 [redacted]

for ejecting a supersonic jet, but for receiving a parallel flow of gas at high supersonic speed in order to compress and arrest the incoming gas.

More specifically, the contents of the present report can be summarized as follows:

<u>Part I</u> contains a three-dimensional treatment of flow through nozzles as a refinement of the customary one-dimensional theory, under the following assumptions: 1) The fluid is ideal and homogeneous, and the flow isentropic, steady and irrotational. 2) Viscosity and heat conduction are ignored. 3) It is further assumed that shocks and jet detachment do not occur in the nozzle; however, conditions for the absence of these phenomena are analyzed.

A simple formula for the thrust produced by such a flow through a nozzle is derived and applied to various types of contours; particular emphasis is given to widely divergent short nozzles. Among families of nozzles yielding the same thrust, the shortest nozzle is determined. Tables I to III at the end of Part I (p. 35) show numerical results in a condensed form.

<u>Part II</u> concerns, first, the construction of "perfect" exhaust nozzles which yield maximum thrust and a parallel

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exhaust flow for a prescribed expansion ratio. Secondly, reverse nozzle flow or "compressor" flow is discussed. "Perfection", which is of little importance for exhaust flow, is here essential.

The stability of the compressor flow is then discussed; it is shown that shocks must be admitted to avoid instability of the phenomena under slight variations of velocities and pressures.

Furthermore, conditions are analyzed for the possibility of a supersonic stream of air entering a compressor without interference of shocks. Limitations of the rim angle are given for a compressor carried in the nose part of a projectile.

Finally, some remarks on the drag against a compressor carrying projectile are added.

The Appendix supplies mathematical detail for Part I. Furthermore, it contains an exposition of the mathematical procedure for the construction of perfect nozzles; the computational schemes developed in the Appendix can easily be applied to other problems of a similar character.

R. Courant

Contractor's Technical Representative

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THEORETICAL STUDIES ON THE FLOW THROUGH NOZZLES AND RELATED PROBLEMS

Nozzles of customary design, ending in a cone of small angle of opening, have proved to be very satisfactory for the purpose of rocket propulsion. Thrusts between 90 percent and 100 percent of the value predicted by gas dynamical theory were found in a great number of tests. Also, when the nozzle was somewhat shortened or lengthened or if the angle of opening was varied by several degrees, the resulting thrust remained quite satisfactory. This means that not much increase in efficiency can be expected from refinements in nozzle construction.

There are situations, however, calling for deviation from conventional nozzle design. In particular it may be desired to make the nozzle rather short and accordingly the angle of opening rather wide. For such nozzles the usual "one-dimensional" or "hydraulic" theory is no longer adequate and a more refined treatment is necessary to obtain theoretical information. Such a refinement of the hydraulic treatment is developed in the first part of this memorandum. Simple formulas are derived for pressure distribution and thrust. These formulas are more accurate than those derived from the

hydraulic theory and are therefore applicable to shorter nozzles with somewhat wider angles of opening.

According to the hydraulic theory, the thrust produced by the flow through a nozzle depends only on the expansion ratio, i.e., the ratio of exit to throat cross-section area; thus short nozzles would yield the same thrust as long ones if their expansion ratio is the same. One cannot expect this conclusion to be correct, and indeed the refined theory shows that the optimal thrust can be obtained only by very long nozzles. Nevertheless, it remains true that one can obtain a few percent less than the optimal thrust by much shorter nozzles.

To obtain the optimal thrust requires a particular nozzle design which we shall refer to as "perfect". The end section of such a perfect nozzle bends gently inward so as to produce a completely axial thrust flow with constant velocity. While it is of no practical importance to make the nozzles perfect if their purpose is to create thrust, it is necessary to design perfect nozzles for other problems in which straightness and constancy of the exhaust flow are more relevant than shortness. Apparently this is the case for supersonic wind tunnels and for supersonic diffusers or

compressors. A compressor is a perfect nozzle to be operated in reverse direction. The air is to enter the compressor with parallel and constant supersonic velocity and is to be compressed past the throat, possibly so as to reduce the velocity to zero.

Of particular interest are supersonic compressors which are carried by a projectile-shaped body opposed to a uniform stream of air. The problem arises of determining under which circumstances the air would enter the compressor with constant and parallel velocity and without interference of shocks.

A further problem is that of the stability of the flow through a compressor. It will be shown that the "perfect" shockless flow is unstable while the flow appears to be stable when certain shock fronts in the interior of the compressor are admitted.

The problems of the perfect nozzle and of the supersonic compressor are discussed in the second part of this memorandum.

Detailed mathematical developments are placed in the Appendix.

PART I.

ISENTROPIC FLOW THROUGH NOZZLES

1. Basic Relations and Hydraulic Treatment.

The nozzle to be investigated is a "de Laval" nozzle; i.e., it is of the "converging-diverging" type and has axial symmetry. The gas is to leave the chamber (c) with velocity zero, to become supersonic while passing through the throat (t), and to enter the atmosphere when leaving the nozzle exit (e). Cf. Fig. 1.

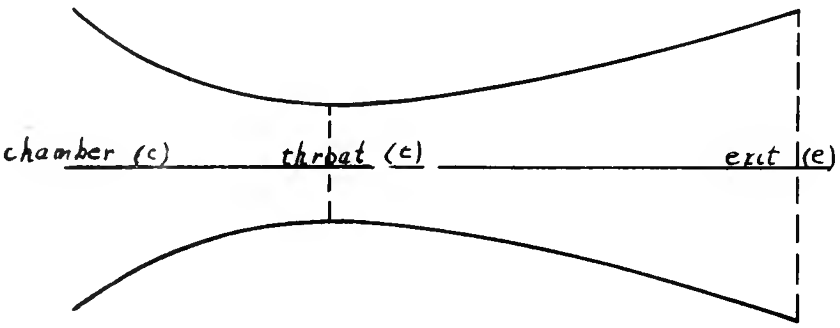

Fig. I

We assume the gas to be ideal, homogeneous(*) and isentropic and the flow to be steady and irrotational. Viscosity and

---

* For exhaust gases of a rocket motor this assumption implies that the fuel has burnt completely in the chamber; if the gas is not homogeneous, one may employ an average adiabatic exponent to some approximation.

 [redacted]

heat conduction are ignored. This appears to be justified for the exhaust flow (Cf. T. Stanton [1])(*) except that viscosity will play a role in determining where jet detachment and shocks occur. The formulas to be given are valid as long as the flow remains free of shocks and as long as the jet remains attached to the nozzle wall.

The isentropic character of the gas flow is expressed by the relation

$$p \rho^{-\gamma} = \text{const.} \tag{1.01}$$

p being the pressure, ρ the density (mass per unit volume) and γ the adiabatic exponent. The sound velocity

$$c = (\gamma p \rho^{-1})^{1/2} \tag{1.02}$$

is connected with the flow speed q through Bernoulli's law which we write in the form

$$(1 - \mu^2) c^2 + \mu^2 q^2 = c_*^2 = q_*^2 \tag{1.03}$$

---

* Numbers in brackets refer to the bibliography at the end of this memorandum.

[redacted]

with the abbreviation

$$\mu^2 = \frac{\gamma - 1}{\gamma + 1}$$

The significance of the "critical" speed $q_* = c_*$ is that the flow speed is critical when it coincides with the local sound speed. Subsonic and supersonic flow can then be distinguished by $q < q_*$ and $q > q_*$ since, as is easily seen, $q \lessgtr q_*$ implies $q \lessgtr c$.

With the aid of Bernoulli's law one can express the quantities c, p, ρ in terms of q/q_*; using the abbreviation

$$\nu = \frac{1}{\gamma - 1}$$

$$(1.04) \qquad \left(\frac{c}{c_*}\right)^2 = \frac{1 - \mu^2 (q/q_*)^2}{1 - \mu^2}, \quad (c_* = q_*)$$

$$(1.05) \qquad \frac{p}{p_*} = \left(\frac{c}{c_*}\right)^{2\nu + 2} = \left[\frac{1 - \mu^2 (q/q_*)^2}{1 - \mu^2}\right]^{\nu + 1}$$

$$(1.06) \qquad \frac{\rho}{\rho_*} = \left(\frac{c}{c_*}\right)^{2\nu} = \left[\frac{1 - \mu^2 (q/q_*)^2}{1 - \mu^2}\right]^{\nu}.$$

The "one-dimensional" or hydraulic theory of O. Reynolds (1885) assumes, as a first approximation, that the flow speed q and hence also c, p, ρ are constant over cross-sections perpendicular to the nozzle axis. The mass flux per unit time across a cross-section of area A is then given by $A \rho q$; hence this quantity is a constant, or

$$\frac{A}{A_*} = \frac{\rho_* q_*}{\rho q}$$

A_* being the area of the cross-section at which the critical speed is assumed. One easily derives the relation

$$\frac{d\ \rho q}{\rho q} = (1 - M^2)\frac{dq}{q}, \tag{1.07}$$

with

$$M^2 - 1 = (q/c)^2 - 1 = \frac{q^2 - q_*^2}{q_*^2 - \mu^2 q^2},$$

($M = q/c$ being the Mach number) from which follows the hydraulic approximation,

$$\frac{dA}{A} = (M^2 - 1)\frac{dq}{q}. \tag{1.08}$$

This relation shows that a flow with increasing speed q is possible for $q < q_*$ or $q < c$ only when A decreases, and for $q > q_*$ or $q > c$ only when A increases. In other words, in the hydraulic approximation, flow with increasing speed is possible only when the critical speed is just reached at the throat.

2. Refined Treatment

In order to refine the hydraulic treatment just explained it is necessary to employ the basic partial differential equations for an irrotational isentropic flow. We introduce coordinates: x, along the axis in the exhaust direction, and y, the distance from this axis; further we introduce the angle θ of the flow direction with respect to the x-axis, the potential function ϕ and the stream function ψ.

The stream function ψ is to be so normed that $\pi\psi^2$ is the rate of mass flow per unit time carried by the stream tube $\psi \leq$ const. The basic equations for the four dependent variables ϕ, ψ, q, and θ can then be written in the form

(2.01) $$\frac{\partial \phi}{\partial x} = q \cos\theta \ , \ \frac{\partial \phi}{\partial y} = q \sin\theta$$

 [redacted]

(2.02) $\psi \dfrac{\partial \psi}{\partial x} = - \rho q y \sin\theta \;, \; \psi \dfrac{\partial \psi}{\partial y} = \rho q y \cos\theta \; .$

The density ρ is to be expressed in terms of q by means of relations (1.06) and (1.04).

It would be natural to prescribe nozzle contour and state in the chamber and to ask for the resulting flow. Mathematically this would mean solving a boundary value problem for equations (2.01) and (2.02). To simplify the task we reverse the procedure: We first prescribe the velocity distribution along the axis.

(2.03) $\qquad q = q_0(x) \text{ for } y = 0$

and then determine possible nozzle contours from the resulting stream surfaces. To this end we introduce the stream functions ϕ and ψ as independent variables instead of x and y, consider the quantities x, y, θ, and q as dependent variables, and develop them with respect to powers of ψ.(*) It is sufficient here to report

---

* One may also introduce q and ψ as independent variables; this might seem to offer advantages since the non-linearity would then refer only to the independent variable q. Nevertheless it was found that the present scheme is better, at least for the expansions up to the second order as given here.

[redacted]

only the results. Details will be carried out in the Appendix, Section 12.

To formulate the results we introduce the important dimensionless quantity h by

$$h = \sqrt{\rho_* q_* / \rho q} \quad , \tag{2.04}$$

the significance of which is that $h^2 = A/A_*$ in hydraulic approximation. Since by (1.06) the density ρ is a given function of q the same is true for h; we have $h = 1$ for $q = q_*$. From (1.07) we obtain

$$2 \, dh/h = (M^2 - 1) \, dq/q \; . \tag{2.05}$$

On the axis, where we had assumed $q = q_o(x)$ the quantity h becomes also a function of x

$$h = h_o(x) = (q_*/q_o)^{1/2} \left[\frac{1 - \mu^2 (q_o/q_*)^2}{1 - \mu^2} \right]^{-\frac{\nu}{2}} \tag{2.06}$$

[Cf. (2.04), (1.06), (1.04)] which is given inasmuch as $q_o(x)$ is given. Also M, defined by (1.08), becomes a given function $M_o(x)$ along the axis when $q = q_o(x)$ is inserted.

We place the origin $x = 0$, $y = 0$, at the throat; more precisely, we place it such that

$$q_o(0) = q_* \tag{2.07}$$

or what is the same thing, such that $h_o'(0) = 0$. We then have

$$h_o(0) = 1, \text{ and } M_o(0) = 1 \,. \tag{2.08}$$

Instead of ϕ and ψ , two other parameters, ξ and η , are introduced by the relations

$$\phi = \int_0^\xi q_o(x)dx \tag{2.09}$$

$$\psi = \sqrt{\rho_* q_*}\,\eta \,. \tag{2.10}$$

Clearly, ξ and η have the dimension of a length and ξ reduces to x on the axis $\psi = 0$, ($\phi = 0$ for $x = y = 0$ being assumed).

In terms of the quantities just introduced, the expansions of x, y, q, θ , h and p with respect to powers of η (instead of ψ) are, if $h_o(\xi)$, $h_o'(\xi)$, $h_o''(\xi)$, $M_o(\xi)$, and $p_o(\xi)$ are written simply h_o, h_o', h_o'', M_o, and p_o respectively:

[redacted]

(2.11) $x = \xi - \frac{1}{2} h_o h_o' \eta^2$

(2.12) $y = h_o \eta \left\{ 1 + \frac{1}{8} [(M_o^2 - 1) h_o h_o'' - (h_o')^2] \eta^2 \right\}$

(2.13) $q = q_o \left\{ 1 + \frac{1}{2} h_o h_o'' \eta^2 \right\}$

(2.14) $\theta = h_o' \eta$

(2.15) $h = h_o \left\{ 1 + \frac{1}{4} (M_o^2 - 1) h_o h_o'' \eta^2 \right\}$

(2.16) $p = p_o \left\{ 1 - \frac{\gamma}{2} M_o^2 h_o h_o'' \eta^2 \right\}$

(For a detailed derivation see Section 12).

It is to be noted that the surfaces ξ = const. and η = const. are the potential and stream surfaces respectively. Thus the two equations (2.11) and (2.12) yield parametric representations for potential and stream surfaces. If $q_o(\xi)$ is chosen, and then $h_o(\xi)$ and $M_o(\xi)$ are determined from (2.05) and (1.08), these stream and potential surfaces are easily drawn. Each stream surface may serve as nozzle contour.

If the terms involving η^2 were neglected one would obtain $x = \xi$, $y = h_o(\xi)\eta$, or $y = h_o(x)\eta$. This

[redacted]

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relation is identical with the result of the hydraulic theory; indeed, since $x = 0$ corresponds to the throat and $h_o(0) = 1$ we would have $A/A_* = (y/y_*)^2 = h_o^2(x)$. It is thus clear that the formulas above represent a refinement of the hydraulic theory.

The relation (2.13) for q yields an improvement over the hydraulic assumption that the speed is constant on the potential surfaces. To form an idea about the magnitude of the deviation from constancy we may introduce the radius of curvature R of the stream contour $\eta = \text{const}$. R is obtained approximately from $y = h_o(\xi)\eta$, $x = \xi$, as

$$R = 1/h_o''\eta.$$

Therefore, relation (2.13) can be written in the approximate form

$$q = q_o\left(1 + \frac{1}{2}\frac{y}{R}\right). \tag{2.17}$$

This formula shows that the deviation of the speed from that at the axis depends on the ratio of the width of the nozzle to the radius of curvature of the nozzle contour.

Unless the end section of the nozzle curves inward as would be the case for $h_o''(\xi) < 0$, the speed increases

[redacted]

and thus, on moving away from the axis, the pressure decreases along the potential surfaces. This is equivalent to saying that the lines of constant speed and hence the lines of constant pressure bend backwards as shown in Table I. Our formulas thus indicate that the <u>pressure along the nozzle wall assumes values that are less than those calculated from the hydraulic theory</u>. This behavior could be expected since a widely divergent opening will give the gas an opportunity for quick expansion.

3. <u>Critical Remarks</u>

Before deriving formulas for the thrust and before considering examples, we shall discuss several possible objections to the preceding method.

On <u>mathematical grounds</u> it may be considered improper to prescribe a quantity such as $q_o(x)$ along the axis, as we did, whereas the proper procedure would have been to prescribe the nozzle contour and the state in the chamber, in agreement with what is prescribed in reality. A way of prescribing data mathematically is said to be proper if for such data there exists a unique solution, which depends continuously

on these data.(*) Prescribing the quantity $q_o(x)$ on the axis is indeed not proper in this sense. It is true, if $q_o(x)$ is an analytic function, a unique solution of the differential equations exists in the neighborhood of the axis but one does not know how far this solution extends without developing singularities. It is also known that if $q_o(x)$ is analytic the expansion employed converges in a neighborhood of the axis but one does not know how large the range of convergence is. Further it is known that slight changes in the character of the function $q_o(x)$ may cause considerable changes in the solutions not only in the subsonic range but in the supersonic range as well. This will, however, not occur if the change in $q_o(x)$ is "<u>smooth</u>" and <u>analytic</u>. Our method, therefore, seems to be justified as long as the function $q_o(x)$ is chosen <u>smooth</u> and <u>analytic</u>. The same character is then imparted to the streamlines in the neighborhood of the axis. Thus we can say: Our method for solving the differential equations of irrotational isentropic flow (2.01), (2.02), is <u>applicable if the desired nozzle</u>

---

* For this notion see [2] p. 171-179. That prescribing nozzle contours and state in the chamber is proper mathematically in this sense is likely, though not completely proved.

contour is represented by a smooth and analytic curve. However, the method cannot be expected to be applicable if the nozzle is non-analytic; e.g., if it is pieced together of analytic sections. (The approximations given so far probably remain valid if the curvature of the nozzle contour remains continuous).

More serious objections can be raised on physical grounds. It is doubtful whether for nozzles with wider divergence the assumption can be upheld that viscosity can be ignored, or that the flow is steady and remains attached to the wall. In this connection it is interesting to confront the result of our calculations with the experimental results of R. P. Fraser [3].

Fraser has tested the flow of compressed air into the atmosphere through a variety of nozzles each having the same expansion ratio (i.e., exit area to throat area), six to one, but with various half-angles of divergence. He has employed chamber pressures ranging from 80 to 13 atmospheres. In addition he has made shadowgraphs of the emerging jet and has measured the pressure at the wall near the exit. For 5° and 10° half-angles of divergence, the pressure agrees rather well with that derived from the hydraulic theory, but for 15° divergence the

 [redacted]

pressure is about 5 to 8 percent less, while for 20°, $22\text{-}1/2^\circ$, 25°, or 30° divergence, the pressure is 10 to 15 percent greater than it would be according to the hydraulic theory. The latter result contradicts the expectation that the pressure at the wall would be reduced when the divergence increases.

Fraser's explanation of the increase of wall pressure for wider divergences is that "it may be that the frictional loss in the convergent portion is the more important fact. This causes, in the pressure at the throat, a consequent drop to below the theoretical critical pressure and, therefore, a consequent drop in the pressures along the divergence. The rising mouth pressure with increasing divergence must then be attributed to a further departure from the adiabatic expansion in the divergence due to general turbulence rather than wall friction."(*) There is, however, another point which is important and probably decisive for the interpretation of Fraser's results. The contours of the tested nozzles

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* A similar opinion is expressed in [4]. "It will be seen that the losses are small and sensibly constant for nozzles with divergence less than 10°, while they are doubled when the cone angle changes to 15°. Since in the latter case the length of the nozzle is less and there is likely to be more separation (and so less surface friction) it would seem that the effects of turbulence become more pronounced as the divergence of the nozzle increases."

[redacted]

consist of a circular entry section while the divergent section issuing from the throat is straight, so that the contour suffers a sudden change in direction at the throat. Certainly this change in direction causes a disturbance which should be considerable for wider divergences. It is hard to say of what this disturbance consists. The flow will probably shoot out of the throat as out of an orifice but later re-attach to the nozzle wall. In any case a shock front will result from this disturbance as is clearly visible in most of the photographs of the emerging jet. Since the gas suffers a sudden increase in pressure when it crosses the shock front, it is possible that this shock is mainly responsible for the major portion of the 10 to 15 percent increase in pressure observed.

4. Remarks on Jet Detachment

Jet detachment within the nozzle has frequently been observed to occur when the chamber pressure p_c is so low that the exit pressure that would result from undisturbed expansion would be noticeably less than the outside pressure p_a (here assumed to be 1 atm). In breaking away from the wall the jet will in general change its direction, this change being accomplished by an oblique shock front beginning

at the point of detachment. See Fig. 2. (One might expect

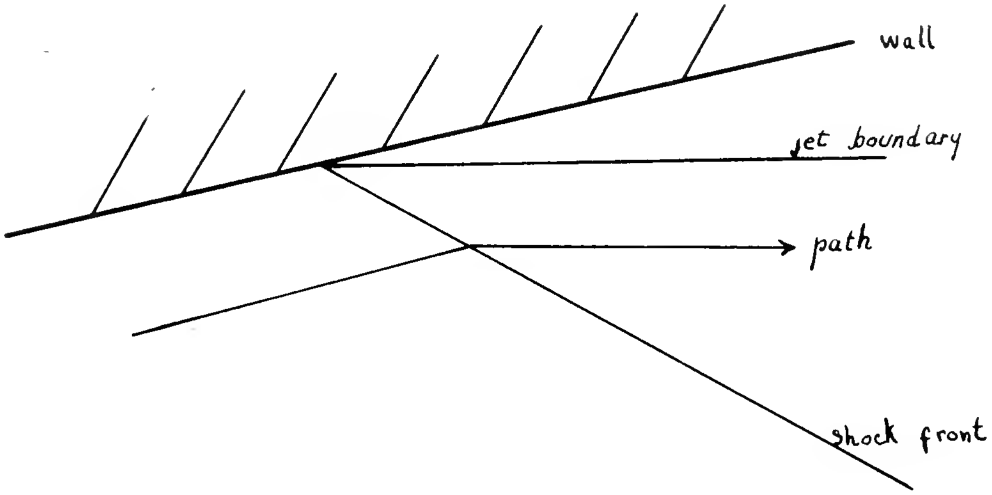

Fig. 2

a state of rest beyond the detached jet, but Stanton's experiments indicate back flow and pressure rise there). Under what circumstances jet detachment occurs depends, as A. Busemann([5] pp. 395,399) explains, upon the stability of the resulting flow pattern which is essentially determined by the boundary layer at the wall. This stability may be very feeble as certain experiments of A. Stodola ([6] end of section 33) indicate. Strong dependence on Reynolds number is indicated by experiments of Stanton ([1] p. 324 Fig. 8) who has tested (for low pressure ratios p_c/p_a) three geometrically similar

nozzles and also varied the chamber temperature. These and other experiments indicate that the point of detachment recedes toward the throat when the chamber pressure p_c (or the ratio p_c/p_a) is lowered. Fraser finds in his experiments with nozzles of various divergence that jet detachment just begins when the exit pressure has been reduced (by reducing the chamber pressure) to about .5 to .65 atmospheres. It is remarkable that this should hold rather independently of the length of the nozzle. One might have expected that for a widely divergent short nozzle the jet would have broken away from the wall without change in direction as soon as it reached outside pressure.

In any case, it appears that the jet will not detach from the wall before it has reached outside pressure. To avoid jet detachment one should, therefore, make the nozzle so short that the pressure at the nozzle wall remains above outside pressure; this is also of advantage for producing a larger thrust, (see the following section). For a widely diverging nozzle the hydraulic theory will in certain cases predict a value of the exit pressure above atmospheric pressure while our refined formulas indicate expansion below atmospheric pressure and thus the possibility of jet detachment.

5. <u>Thrust</u>

For the reactive <u>thrust</u> produced by the flow through the nozzle, one can easily derive a simple approximate formula which is more exact than that given by the hydraulic theory. It is customary to define the total thrust F as the difference

$$F = F_i - F_a \tag{5.01}$$

of the internal thrust F_i, resulting from the pressure acting against the wall of the chamber and the nozzle, and the external counter-thrust F_a that would result if atmospheric pressure were acting against the outer surface of the body in which the nozzle is imbedded. (The contributions to the total force exerted against the body due to the deviation of the external pressure from atmospheric pressure are considered part of the drag and are left to a separate investigation).

To evaluate the thrust we may consider the potential surface S_e: $\phi = \phi_e =$ const. through the exit rim of the nozzle (Cf. Fig. 3) and observe that the internal thrust F_i (counted positive when acting in the negative direction, i.e., against the stream) is

equal to the sum of the axial component of the momentum M_e

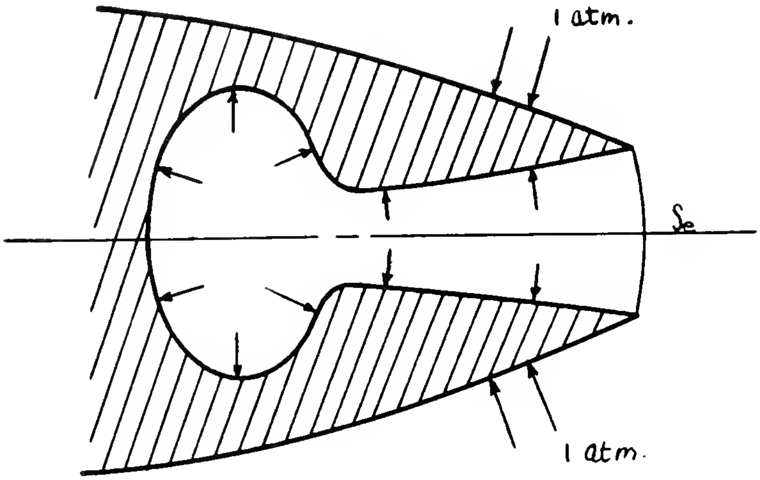

Fig. 3

transported through S_e to the outside per unit time and of the resultant pressure force, P_e, exerted against the surface S_e from the inside.(*)

Since $\pi \psi^2$ is the mass per unit time flowing through the stream tube $\psi \leq$ const. and πy^2 is the area of the projection perpendicular to the axis of that section of S_e which is cut out by this stream

---

* I.e. the total pressure force, F_1-P_e, exerted against the volume of gas inclosed by chamber, nozzle and surface S_e, equals the momentum transported per unit time, M_e.

[redacted]

tube, we have

$$M_e = \pi \int_{S_e} q \cos\theta \, d\psi^2$$

and

$$P_e = \pi \int_{S_e} p dy^2.$$

Thus we arrive at the expression

$$(5.02) \qquad F_1 = 2\pi \int_0^{\psi} q \cos\theta \; \psi \, d\psi + 2\pi \int_0^{y} p y dy$$

the integration being extended over the surface S_e.

The external counter-thrust is obviously given by

$$(5.03) \qquad F_a - p_a A$$

p_a being the atmospheric pressure and

$$(5.04) \qquad A = \pi y^2$$

being the area of the projection of the surface S_e on a plane perpendicular to the axis. It is convenient to introduce the dimensionless thrust ratios

$$(5.05) \qquad K_1 = F_1 / A_* p_* , \quad K_a = F_a / A_* p_* , \quad K = K_1 - K_a$$

[redacted]

where A_* is defined by

$$(5.06) \qquad A_* = \pi \eta^2 = \pi \psi^2 / \rho_* q_* \ .$$

The significance of A_* is that it is the throat cross-sectional area in hydraulic approximation (Cf. Sec. 1).

Introducing the variable η (Cf.(2.04)) along the surface S_θ one can easily derive (Cf. Sec. 12) the expressions

$$(5.07) \qquad K_1 = \frac{2}{(1-\mu^2)\eta^2} \int_0^\eta (q/q_* + q_*/q) \cos\theta \ \eta d\eta$$

$$(5.08) \qquad K_a = (p_a/p_*)(A/A_*) = (p_a/p_*)(y/\eta)^2 \ .$$

To evaluate these quantities one may expand them with respect to powers of η and retain terms up to the third order. Instead of using the resulting expressions, it seemed preferable to use different ones which are much simpler expressions and which coincide with the exact ones up to terms of the third order in η (Cf. Sec. 12). To this end one introduces the averages $\tilde{q}$ and $\widetilde{h^2}$ of q and h^2 over the surface S_θ;

$$(5.09) \qquad \tilde{q} = \tfrac{1}{2}(q_o + q) = q_o[1 + \tfrac{1}{4} h_o h_o'' \eta^2]$$

 [redacted]

and

$$(5.10) \qquad \tilde{h}^2 = \tfrac{1}{2}\,(h_o^2 + h^2) = h_o^2\,[1 + \tfrac{1}{4}(M_o^2 - 1)\; h_o h_o''\; \eta^2]$$

derived from (2.13) and (2.15). Here q and h^2 refer to the nozzle exit rim. Further one needs the expression for the half-angle of opening at the nozzle rim. This is given, in first order, by:

$$\theta = h_o' \eta$$

as is derived from (2.11) and (2.12).

Employing these average quantities one has (up to terms of third order in η)

$$(5.11) \quad K_1 = \frac{1}{1-\mu^2}\,(\tilde{q}/q_* + q_*/\tilde{q})\,\cos^2(\theta/2),$$

$$(5.12) \quad K_a = (p_a/p_*)\,\tilde{h}^2\,\cos^2(\theta/2)$$

$$(5.13) \quad K = \left[\frac{1}{1-\mu^2}\,(\tilde{q}/q_* + q_*/\tilde{q}) - (p_a/p_*)\tilde{h}^2\right]\cos^2(\theta/2)\,.$$

This approximate expression for the thrust coefficient is similar in form to that of the hydraulic theory,

$$K = \frac{1}{1-\mu^2}\,(q_o/q_* + q_*/q_o) - (p_a/p_*)h_o^2\,,$$

[redacted]

and is moreover exact for purely radial flow through a conical nozzle.(*)

The hydraulic expression for K would be approached by making the nozzle infinitely long; the

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* To see this let the potential surface S_e be the sphere $r = \text{const.}$ Then, in view of $q = q_o$ and $h = h_o$ on S_e, one has

$$\eta^2 = 2r^2(1 - \cos\theta)h_o^{-2}$$

on S_e and

$$\eta^{-2}\int_0^{\eta} \cos\theta \, d\eta^2 = (1 - \cos\theta)^{-1}\int_0^{\theta} \cos\theta \sin\theta d\theta = \cos^2(\theta/2),$$

hence, from (3.07),

$$K_i = \frac{1}{1-\mu^2}(q_o/q_* + q_*/q_o)\cos^2(\theta/2).$$

Further, from $y = r \sin\theta$, one has

$$y^2/\eta^2 = h_o^2 \cancel{\sin^2\theta/2}\,(1 - \cos\theta) = h_o^2 \cos^2(\theta/2),$$

whence

$$K_a = (p_a/p_*)\, h_o^2 \cos^2(\theta/2) .$$

[redacted]

maximum of this expression is easily found to be

(5.14) $$K_{max} = \gamma q_o/q_* \ .$$

Therefore, K/K_{max} may serve as a measure for the efficiency of the exhaust nozzle.

In the design of a rocket exhaust nozzle it is natural to aim at as large a thrust as possible. The problem of <u>maximum thrust</u> may be split into two parts:

1. An indefinitely long nozzle contour is given, which may be cut off at any "end" point. The thrust produced by the remaining section is to be investigated as a function of the end point. The maximum thrust and the corresponding "optimal" section are to be determined. (Cf. Malina [7]).

2. A "perfect" nozzle contour is to be found whose maximum thrust is as large as possible.

Both questions can be answered completely. The answer to the first question is that <u>the thrust attains its maximum when the contour is cut off at the point at which the gas pressure has just been reduced to outside pressure</u>. In that case the thrust is given solely by the momentum transport per unit time, (i.e., by the first term in the expression (5.02)). For the hydraulic

 [redacted]

approximation this fact is well known and it has been frequently assumed tacitly. It should be emphasized, however, that it holds quite generally.(*) This statement can easily be derived from relations (3.07) and (3.08), (Cf. Sec. 12). A very simple proof is also given in an earlier memorandum [8].

As to the second question a rather obvious condition was derived in the memorandum quoted, viz., that the flow direction be axial on a surface of constant flow speed passing through the rim of the nozzle mouth, provided that the pressure on this surface equals the outside pressure. How to construct such "perfect" nozzles of finite length will be discussed in the second part of this memorandum.

The task of constructing a "perfect" exhaust nozzle for the purpose of rocket propulsion is rather academic, since the maximum of the curve representing the thrust as a function of the end point is very flat for such a perfect nozzle contour. The same is true for conventional nozzle contours.

---

* F. Malina, [7], in an attempt to take the divergence into account obtains a result which is not in agreement with the above statement. The reason is that no account is taken of the changes of speed and pressure over the cross-section, although these changes are of the same order of magnitude as the changes in direction considered.

[redacted]

Experiments(*) with more than a hundred tests have shown that the thrust is very insensitive to the length of the nozzle and that its value is between 90 and 100 percent of the value derived from the hydraulic theory. Hence, shortening conventional nozzles entails but little loss in thrust and in addition keeps the pressure well above outside pressure thus ensuring the absence of jet detachment (Cf. Sec. 4).

However, the thrust is more sensitive for more widely divergent nozzles. Therefore, it would be desirable to obtain information about the thrust from the refined approximation formula (3.13); but it seems difficult to deduce specific properties from it. Instead we shall formulate certain general conclusions that were gathered from the evaluation of the thrust in various examples discussed in the following section. The conclusions (of necessity somewhat vaguely formulated) are:

1. The maximum thrust of a contour is the smaller the more divergent the contour is.

2. The maximum thrust obtained from the short "optimal" section of a widely divergent contour is larger than the thrust obtained from an equally short section of a less divergent contour.

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* B. H. Sage [9]

 [redacted]

The latter remark is of significance since it implies, as stated in the introduction, that if one wants to cut off a section of the nozzle, and is willing to sacrifice a few percent of thrust, then one could obtain the same thrust by making the nozzle still shorter but more divergent. It should, of course, be remembered that this statement should be qualified by adding that it holds only if the various assumptions enumerated at the beginning of Section 1 remain valid for more divergent nozzles. According to the discussion in Section 3, it is doubtful whether this is so. If the assumptions are valid then the various arguments discussed are all in favor of shorter nozzles.

6. Examples

As a first example, we consider a set of nozzles which are approximately symmetrical with respect to the throat. These contours (referred to as of type C_1) are, to some approximation, hyperbolas.(Cf. Table I). They are obtained by setting

$$h_o(\xi) = \sqrt{1 + (\xi/e)^2}, \tag{6.01}$$

e being an appropriately chosen length. The function

[redacted]

$q_o(\xi)/q_*$ is then to be found by solving

$$\left[\frac{1-\mu^2(q_o/q_*)^2}{1-\mu^2}\right]^{-\nu}\qquad\frac{q_*}{q_o} = 1+(\xi/e)^2.$$

$M_o(\xi)$ is then determined. With $eh_o' = (\xi/e)\Big/\sqrt{1+(\xi/e)^2}$, $e^2h_o'' = 1\Big/\left(\sqrt{1+(\xi/e)^2}\right)^3$, relations (2.11) and (2.12) become

$$x = \xi\left\{1-\tfrac{1}{2}(\eta/e)^2\right\}$$

(6.02)

$$y = \eta\sqrt{1+(\xi/e)^2}\left\{1+\frac{1}{8}\left[M_o^2-1-(\xi/e)^2\right]\frac{(\eta/e)^2}{1+(\xi/e)^2}\right\}.$$

To first approximation, they agree with the relation

$$x = \xi\sqrt{1-(\eta/e)^2},\quad y = \eta\sqrt{1+(\xi/e)^2};$$

hence streamlines, η = const., and potential lines, ξ = const., are, in first order, confocal hyperbolas and ellipses respectively. In any case the streamlines approach straight lines at both ends, $|\xi| \to \infty$.

The speed q along potential surfaces ξ = const. is given by

(6.03) $$q = q_o(\xi)\left\{1+\tfrac{1}{2}(\eta/e)^2\Big/\left(1+(\xi/e)^2\right)\right\}.$$

[redacted]

Thus the relative deviation of the speed from the value given by the hydraulic theory approaches zero at both ends of the nozzle, i.e., $q/q_0 \rightarrow 1$ as $|\xi| \rightarrow \infty$. A set of streamlines, potential lines, and curves of constant speed, for $\gamma = 1.2$, are shown in Table I. In Table II the same set of streamlines is shown but so magnified that they all intersect the throat diameter in the same point. These streamlines then represent a set of nozzle contours with the same throat diameter. A set of "curves of constant thrust" is shown in this Table, where a pressure ratio $p_c/p_a = 21.28$ is assumed. These curves connect the end points of nozzle contour sections for which the thrust coefficient as calculated by formula (5.13) is the same. For each value of the thrust coefficient K or thrust efficiency K/K_{max}, the shortest contour section is chosen and shown separately in Table IIA. The evidence exhibited in Table II clearly confirms the statements made at the end of Section 5.

The meaning of these statements will become clearer perhaps when contours of type C_1 are compared with an essentially different type of contour. As such we choose for a second example the ones (referred to as of type C_2)

[redacted]

resulting from the assumption

$$h_o(\xi) = 1 + (\xi/c)^2 \tag{6.04}$$

c being an appropriately chosen length. Here we find

$$x = \xi \left\{ 1 - [1 + (\xi/c)^2] (\eta/c)^2 \right\}$$

$$y = \eta \left[1 + (\xi/c)^2 \right] \left\{ 1 + \frac{1}{4} \left[(M_o^2(\xi) - 1)(1 + (\tfrac{\xi}{c})^2) - 2(\tfrac{\xi}{c})^2 \right] (\tfrac{\eta}{c})^2 \right\}.$$

The potential lines are in first approximation the circles passing through the points $x = \pm c, y = 0$; the streamlines form, in first approximation, the orthogonal set of circles. The speed along the potential lines is given by

$$q = q_o(\xi) \left\{ 1 + [1 + (\xi/c)^2] (\eta/c)^2 \right\} . \tag{6.06}$$

For a particular case, a (nearly hyperbolic) contour C_1 and a (nearly circular) contour C_2 are compared in Table III. These curves are chosen such that they have about the same curvatures at the throat. More specifically we have chosen $e^2 = 6$, $c^2 = 11$ respectively. The thrust for every point of the contour, if the contour were broken off there, is plotted.

 [redacted]

Two observations can be made. Firstly, the maximum thrust that can be reached at all is smaller for the more curved contour C_2 than for the more conventional contour C_1. Secondly, the values of the thrust that can be obtained at all for the more curved contour C_2 are obtained from shorter sections than for the less curved contour C_1.

In the two examples considered, the nozzle contours were approximately symmetrical with respect to the throat whereas in actual nozzles, the entry section is usually much more curved than the exhaust section. To obtain information on flow through unsymmetrical nozzles, contours derived from (type C_3)

$$(6.07) \qquad h_o(\xi) = \sqrt{(1+\lambda^2)\,[1+(\xi+\lambda)^2]} - \lambda(\xi+\lambda)$$

were investigated. The slope of the contours approach (approximately) the values $(\sqrt{1+\lambda^2} - \lambda)\eta$ and $-(\sqrt{1+\lambda^2} + \lambda)\eta$ as ξ approaches ∞ and $-\infty$ respectively. The values $\lambda = 1.33$ with $\sqrt{1+\lambda^2} = 1.67$ and $\lambda = 3.05$ with $\sqrt{1+\lambda^2} = 3.20$ were chosen for numerical computation. As a result it was found that the flow past the throat is very much like the flow past the throat for an approximately symmetrical nozzle of

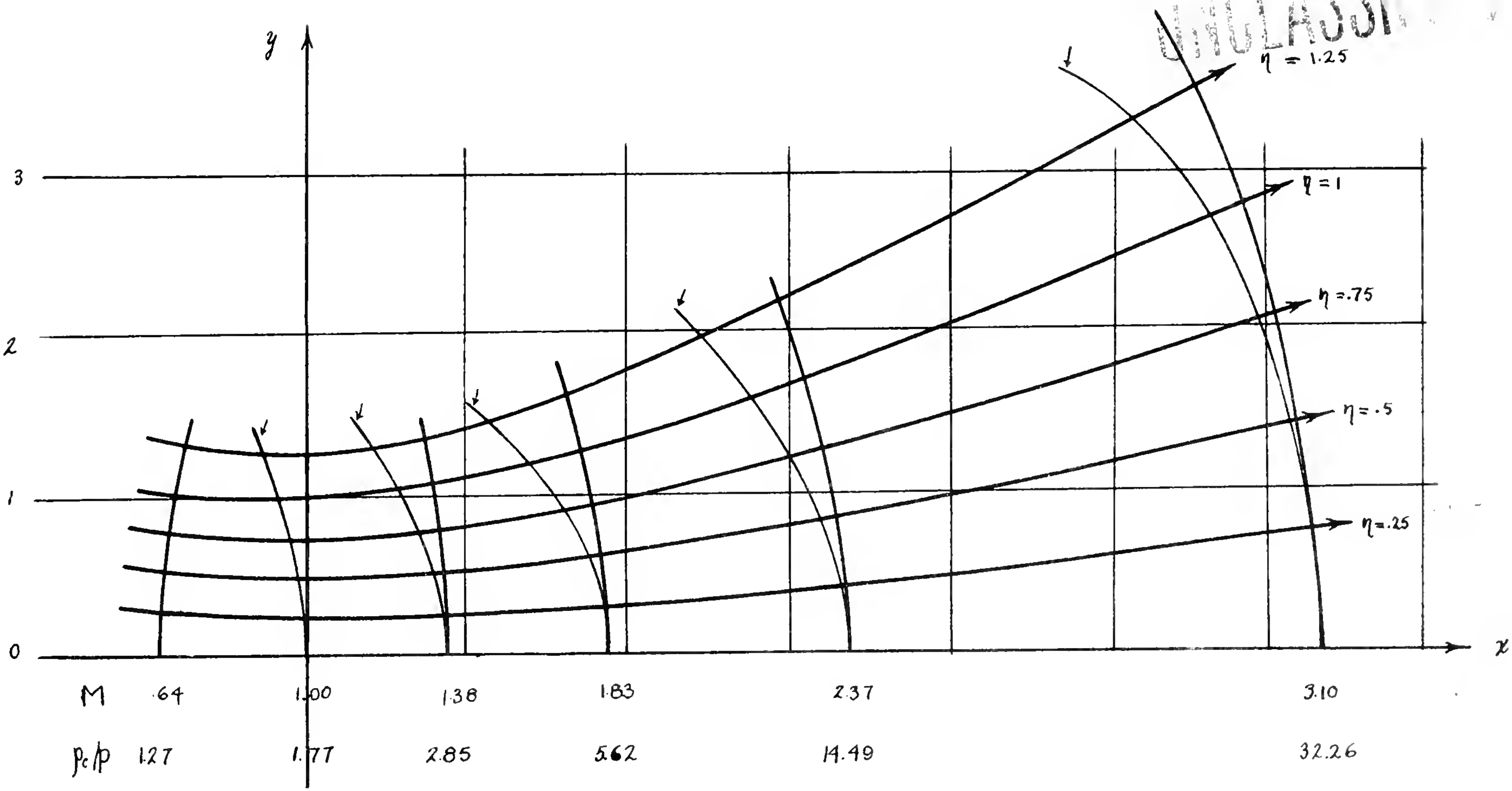
[redacted]

type C_1. One thus gains the impression that the entry section of the nozzle has no great influence on the flow past the throat. It was also found that the convergence of the procedure was far less satisfactory in the subsonic section than in the supersonic section.

Thus the examples discussed in this section offer a justification for the general conclusions formulated at the end of Section 5.

Table I

[redacted]



The figure shows a set of streamlines, potential lines, and lines of constant pressure (marked by small arrows) for exhaust flow C_1. Mach number $M = q/c$ and the pressure p_c/p, indicated along the axis, are constant along the latter lines. The streamlines are characterized by the radius, η, of the throat cross section.

[redacted]

PART II

ON PERFECT EXHAUST NOZZLES AND COMPRESSORS

7. On Perfect Nozzles.

As has been explained earlier (Sec. 5) a nozzle is perfect if it forces the flow to emerge from the nozzle in axial direction with constant speed and a pressure just equal to the outside pressure. The reason is that then the reactive thrust is a maximum, i.e., greater than that resulting from any other nozzle operating with the same ratio of chamber pressure to outside pressure. This statement was qualified; it is valid if the flow is irrotational, steady, isentropic, and if both viscosity and heat conduction can be ignored.

Such a perfect nozzle can be designed without difficulty. As a matter of fact, whenever a diverging exhaust flow is given, it is possible by re-routing only a certain section of it, to make the flow "perfect", i.e., to guide it so that it eventually acquires constant axial velocity. Every streamline of such a perfect flow yields a perfect nozzle. The possibility of constructing a perfect flow was already indicated by Prandtl and

 [redacted]

Busemann.(*)

Perfect nozzles can be constructed so as to produce any desired exhaust velocity; that is, expressed in dimensionless terms, the Mach number of the exhaust flow, or what is equivalent, the ratio of chamber to exhaust pressure can be prescribed. The first step in the construction consists in securing an exhaust flow, the basic "flow" F_o, which leads at least to the desired exhaust velocity. To this end, the method explained in the first part of this memorandum may be employed. The re-routing process for a two-dimensional exhaust flow is so simple that it may be described briefly. First the point A_o on the axis should be found where the basic exhaust flow F_o attains the desired exhaust

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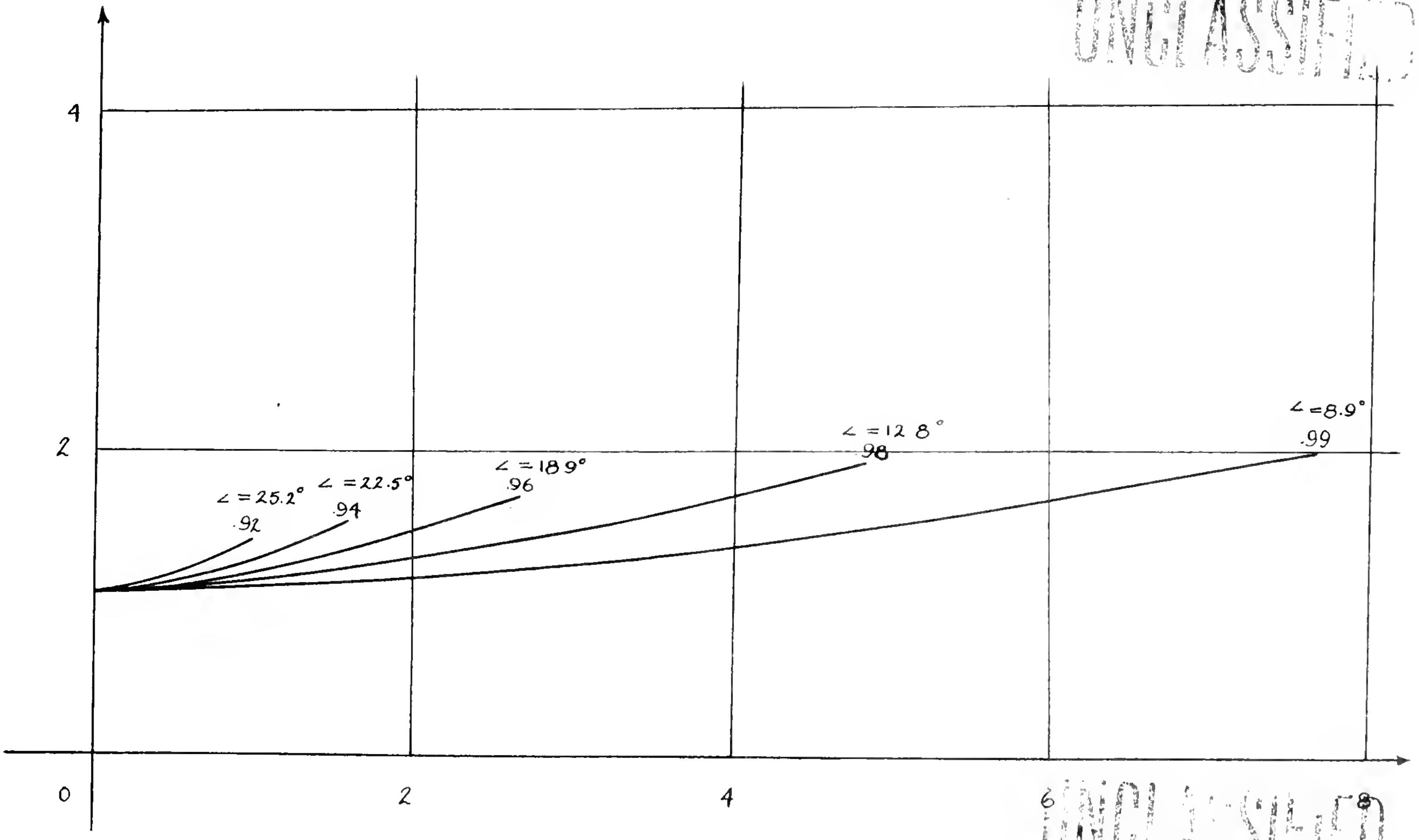
* L. Prandtl and A. Busemann [10] p. 499; Busemann [5] p. 430, and [11] p. 857. In the first two of these papers a graph for such a two-dimensional flow with the Mach number M = 1.85 is shown (Fig. 11 and Fig. 51 respectively). A similar flow picture for M = 1.8 is shown in the third paper (Fig. 10). It is mentioned there that such a construction is also possible for three-dimensional flow with rotational symmetry; a procedure is, however, not described.

Nozzles have been designed on the basis of such a construction (following the two-dimensional pattern) by Mr. Puckett (Aero Dept., California Inst. of Tech.). Experiments confirmed that these nozzles were shock-free, (although their performance otherwise was not noticeably better than that of other nozzles)[12].

[redacted]

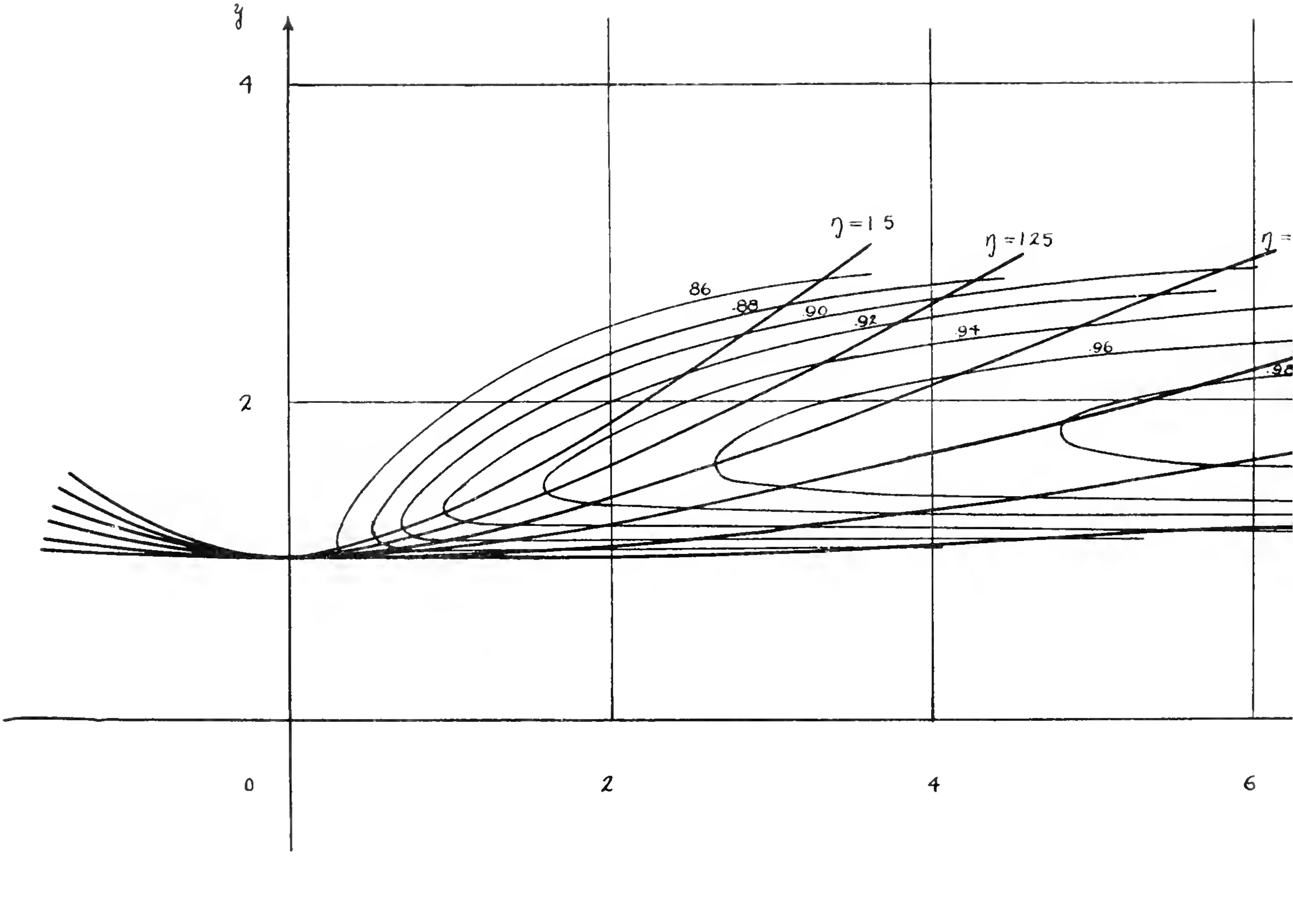
Table IIa

[redacted]



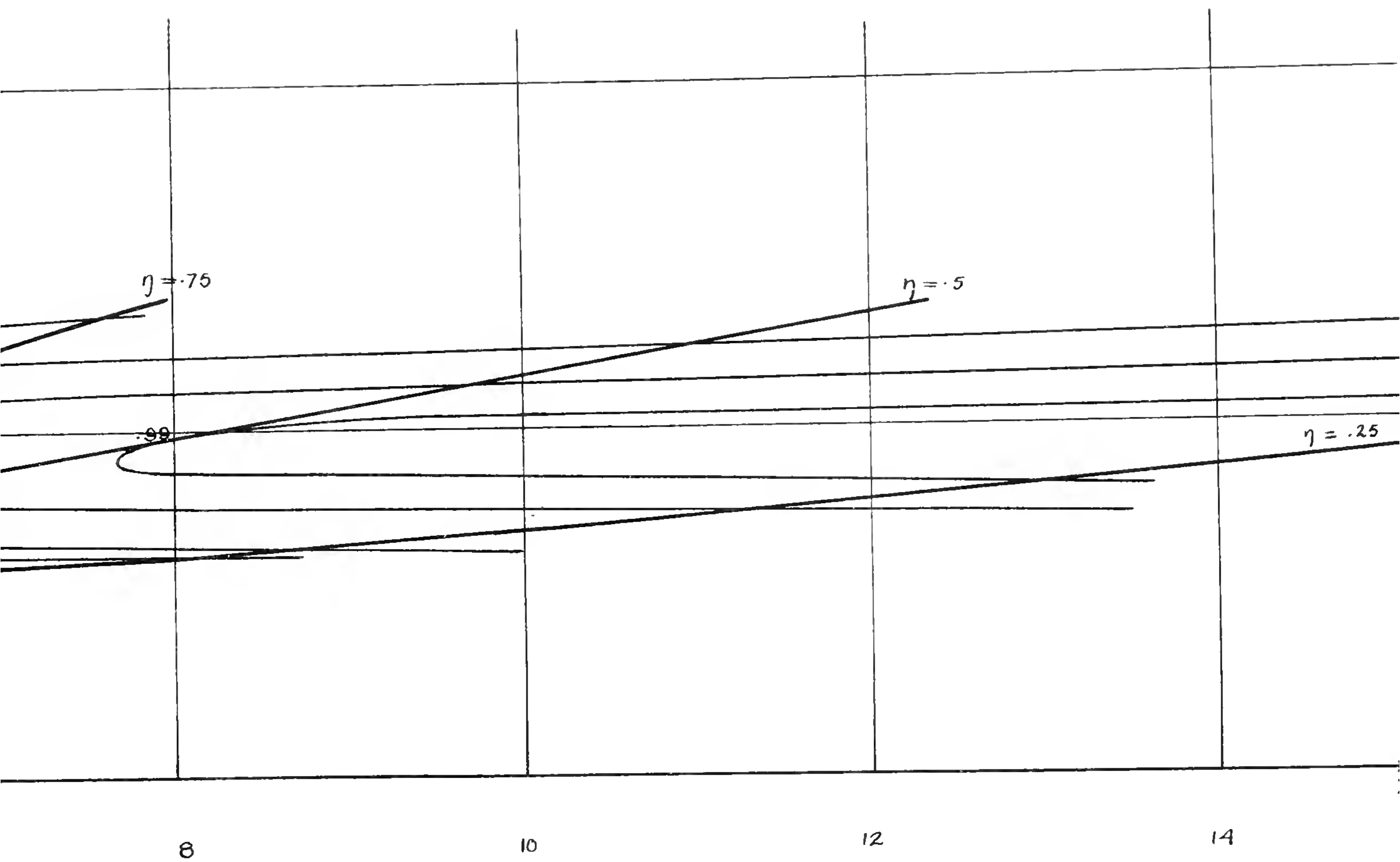
The figure shows a set of shortest contour sections for given values of the thrust ratio, K/F_{max}. They are selected from the contour sections with constant thrust ratio shown in table II. The half-angle of divergence at the rim is indicated.

[redacted]



The figure shows a set of nozzle contours derived by magnifying the streamlines of flow C_1. End points of sections of nozzle contours which yield the same thrust are connected by a line. The theoretical ratio, K/K_{max}, of this thrust

[redacted]

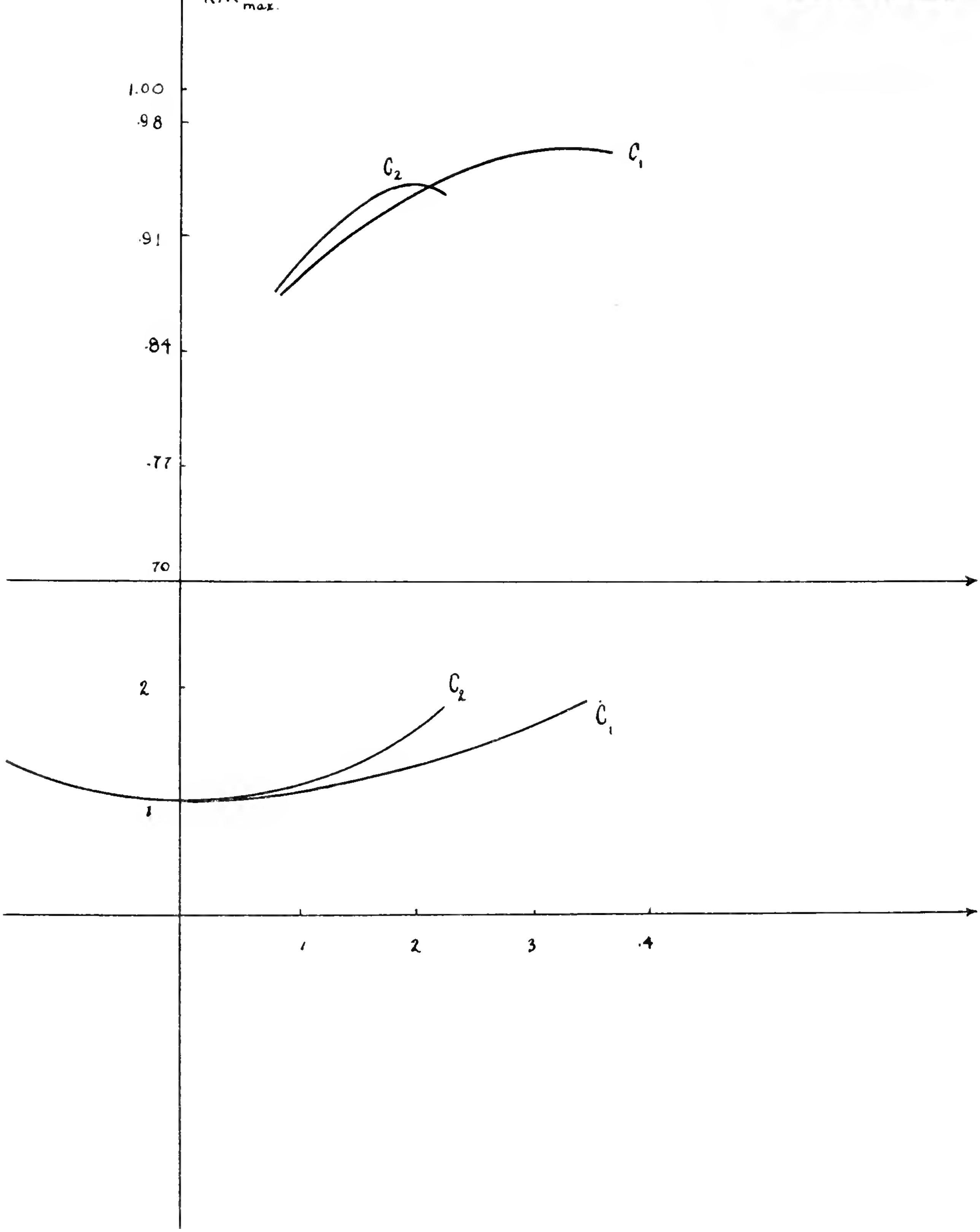


to the maximum thrust, which would result from a "perfect" nozzle, is indicated. The quantity η here equals approximately 6 times the curvature of the contour at the throat (cf. p. 13). [redacted]

[redacted]

[redacted]

Table III

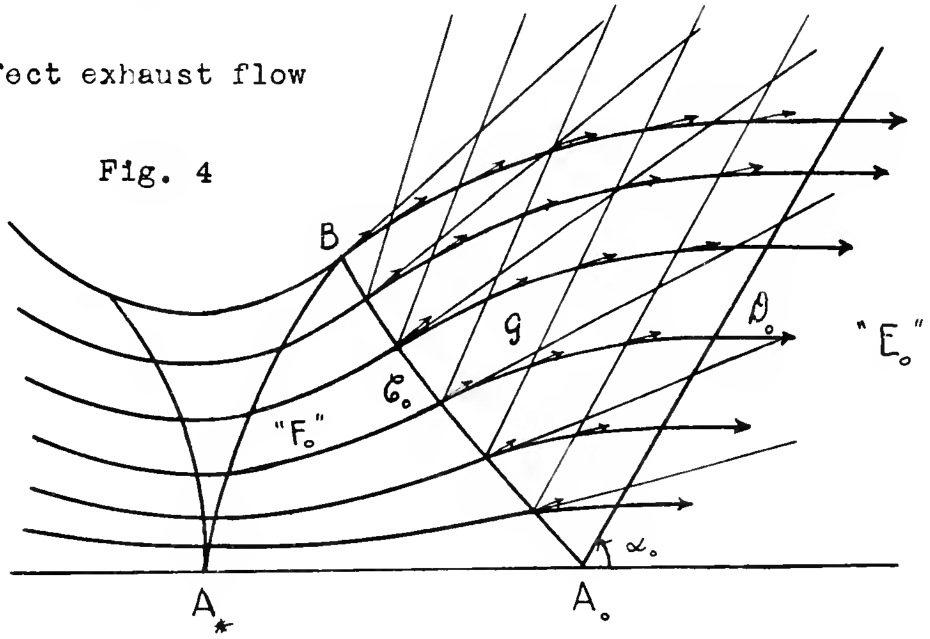


The figure shows two nozzle contours of the type C_1 and C_2 so chosen that their (approximate) curvature at the throat is the same, $1/R = .20$. The thrust ratio K/K_{max} as a function of a variable end point is plotted.

velocity q_o. Through the point A_o two lines are drawn: the backward Mach line C_o which is determined by the flow F_o, and the straight line D_o which would be the forward Mach line of a flow E_o with constant parallel velocity q_o, (pressure p_o and sound speed c_o being determined through the condition of isentropic change). The angle between the line D_o and the axis is just the Mach angle α_o of the flow E_o. Up to the

Perfect exhaust flow

Fig. 4



line C_o the original flow F_o will be retained; in the sector G between C_o and D_o the flow will be changed, and beyond D_o it will be parallel with the constant velocity q_o. To determine the new flow in the sector G, the known directions of the Mach lines of the flow F_o should be marked on the line C_o.

[redacted]

[redacted]

Straight lines, D, should then be drawn from C_c in these directions so as to cover the sector G. Further the known direction of the flow F_o on the line C_o should be marked and each such direction is to be transplanted parallel to itself along the lines D. Thus the directions of the new flow F in the sector G are determined. Through integration of this field of directions, beginning at C_o, the streamlines of the new flow F are obtained. Beyond the line D_o the flow is to be continued with constant axial velocity. We note that the streamlines so constructed suffer a change of curvature on crossing the Mach lines C_o and D_o; these Mach lines will therefore be referred to as "sonic" lines.

There is still the question as to which of the streamlines so constructed should be made the nozzle contour. A reasonable choice can be based on the following considerations:

The described construction is possible as long as the set of Mach lines D do not intersect each other in the sector G. This might happen when one proceeds too far away from the axis; it is certain at least that the angle the Mach lines D make with the axis first increases and then decreases again on moving away from the axis along the Mach line C_o. The place where this angl[redacted]m is easily determined; it is the

intersection B of the backward Mach line C_o with the forward Mach line which begins at the point A_* on the axis at which sound speed is attained. Therefore, one is sure that the construction is possible, as long as the point B of maximum angle is not passed. On the other hand one wants to construct the nozzle as short as possible. Thus one is led naturally to choosing for nozzle contour just the streamline through the point B.[*]

The construction of a perfect three-dimensional flow and a perfect three-dimensional nozzle (with axial symmetry) is similar to that for the two-dimensional flow but not as simple. The Mach lines D are no longer straight (except D_o) and the flow direction is no longer constant along these lines. The flow is to be determined from the basic partial differential equations in the region bounded by the two "sonic fronts" C_o and D_o. These equations can be solved by a numerical procedure rather easily due to the particular type of initial condition involved. An outline of the procedure is described in Section 13.

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* This is apparently the choice Prandtl and Busemann [10] have made according to the figures they show.

Busemann [11] reports about perfect flows determined for various Mach angles. In particular he gives values for the maximum half-angle of divergence along the nozzle contour. (This half-angle is the angle of the flow direction against the axis at the point B in the construction above). This maximum half-angle depends on the Mach number M_o of the exhaust flow. It becomes zero when $M_o \to 1$ and when $M_o \to \infty$; it attains a maximum somewhere between $M_o = 3$ and $M_o = 5$. This maximum is 12° for the two-dimensional flow, and 8° to 9° for the three-dimensional flow, (for air with $\gamma = 1.405$). It would also be interesting to know what the length of these perfect nozzles would be for given throat and exit diameters. Certainly these nozzles would be rather long and very softly curved.

Certain limitations for the curvature of the contour at the exit can be derived; they represent the conditions that no shock develops along the sonic line D_o. Let y be the distance of the exit rim or lip from the axis, and R be the radius of curvature of the contour at the lip. Then for two and three-dimensional flow

[redacted]

$$R/y > \Gamma_o = \frac{\gamma + 1}{2} \Big/ \sin^2 \alpha_o \cos^2 \alpha_o \ ,$$

α_o being the Mach angle, connected with the Mach number M_o of the final flow by the relation $M_o = 1/\sin \alpha_o$. For the derivation see footnote in Section 13, p. 76. For

$$M_o = 3, \quad \gamma = 1.4,$$

we find

$$R/y > 12.15.$$

Also, for

$$M_o = 1.825$$

$$R/y > 5.7,$$

and for

$$M_o = 1.414$$

$$R/y > 4.8.$$

The significance of the latter value is that for $M_o = 1.414 = \sqrt{2}$, the quantity Γ_o attains its minimum. This Mach number is thus most favorable as regards the lip curvature for the two-dimensional flow.

8. Compressor Flow.

Isentropic flow is reversible. Ifa nozzle is so designed that a shockless exhaust flow through it is possible, then, in principle, the reversed flow from exit into chamber is also possible.

[redacted] [redacted]

Thus it seems that all that has been said for exhaust nozzle flow would hold also for compressor flow. It would thus appear to be possible to compress a supersonic stream of air or gas so that it comes to rest by guiding it through a reversed exhaust nozzle, a "compressor". In the following discussion we present arguments which, with certain reservations, speak in favor of the possibility of such an isentropic compressor flow.

Naturally we are interested only in isentropic compressor flow which enters the compressor with constant velocity in axial direction and hence is the reverse of a very special type of exhaust flow, viz., of what was called perfect exhaust flow. Thus the compressor should be designed as the reverse of a perfect exhaust nozzle. While "perfection" was incidental in exhaust nozzle design, it is essential in compressor design.

The exhaust nozzle is designed for a given ratio of chamber pressure p_c to exhaust pressure p_e. On the other hand in compressor flow, speed, pressure, and density at the entrance are prescribed. From these quantities, Bernoulli's Law, and the adiabatic relation, one determines the "stagnation" pressure p_{st} corresponding to vanishing

[redacted] [redacted]

speed. From (1.03) and (1.05)

$$p_{st} = p_e \left[1 + \frac{\gamma - 1}{2} M_e^2 \right]^{\nu + 1} .$$

The stagnation pressure must be maintained in the chamber if the compressor flow is to be isentropic, i.e., $p_c = p_{st}$. Thus the ratio of stagnation pressure to entrance pressure determines the design of the compressor.

The question remains, however, whether the flow into a perfect compressor, operated with the appropriate chamber pressure, will actually follow the reverse of the perfect exhaust flow. One may assert that this is so if the influence of viscosity and heat conduction is negligible. It seems well established that the influence of viscosity on isentropic exhaust flow is very small.(*) One should bear in mind, however, that the action of viscosity is not reversible. While for the exhaust nozzle the rate of growth of the boundary layer developing at the wall is less than for simple parallel flow, this rate of growth is greater for the compressor. Since the flow is directed against a pressure rise there is the danger that the boundary layer detaches somewhere in the diverging section between throat and chamber; but detachment can probably be avoided by keeping the divergence of that section small enough.

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* Except that viscosity does play a role in deciding the position of shock fronts, if they occur. (Cf. Sec. 4, p. 19.)

Other than by detachment, the boundary layer will hardly influence the main stream. In subsonic flow, the boundary layer influences the main stream only by its growth inasmuch as it covers a gradually increasing portion of the main stream; this portion will be very small if Reynolds' number is rather large. In supersonic flow, disturbances in the boundary layer are propagated immediately into the main stream through the Mach lines. It does not appear likely, though, that this influence of the boundary layer in supersonic flow is strong enough to cause strong changes in the main stream. Nevertheless, this question needs further investigation.

Shock fronts are another disturbance in which viscosity is involved. If a compressive flow curves inward too strongly, an isentropic continuation of the flow will not exist and a shock front will develop in the midst of the stream. The shock is, however, eliminated by the perfect design which guarantees the existence of an isentropic continuation as long as the influence of the boundary layer does not interfere.

In expanding flow through an exhaust nozzle, strong shocks can be created by sufficiently increasing the pressure opposed to the stream. This is different for compressive

flow: decrease and not increase of chamber pressure will cause the development of shock front. The reason is that the stagnation pressure decreases when the entropy increases.(*) As will be shown in the following section, it is necessary to operate the compressor with a chamber pressure less than the "ideal" one for which the nozzle was designed, since otherwise the flow would not be stable.

9. The Stability of the Isentropic Compressor Flow.

To investigate the stability of the isentropically compressed flow, one should find out how the flow would change if either the chamber pressure or the entrance air speed (but not the compressor design) were changed. Instead, we ask what steady flows are possible under such changed conditions. The situation is clear for an isentropic exhaust flow. When the chamber pressure is changed,

---

* The stagnation pressure is given by

$$p_{st} = \left[\frac{1+\gamma}{2\gamma} q_*^2 \right]^{\nu + 1} \left[p\rho^{-\gamma} \right]^{-\nu}$$

as seen from (1.02) and (1.03). The critical speed q_* is known to be unchanged across a shock, and $p\rho^{-\gamma}$ depends only on the entropy and increases with increase of entropy.

[redacted]

[redacted] [redacted]

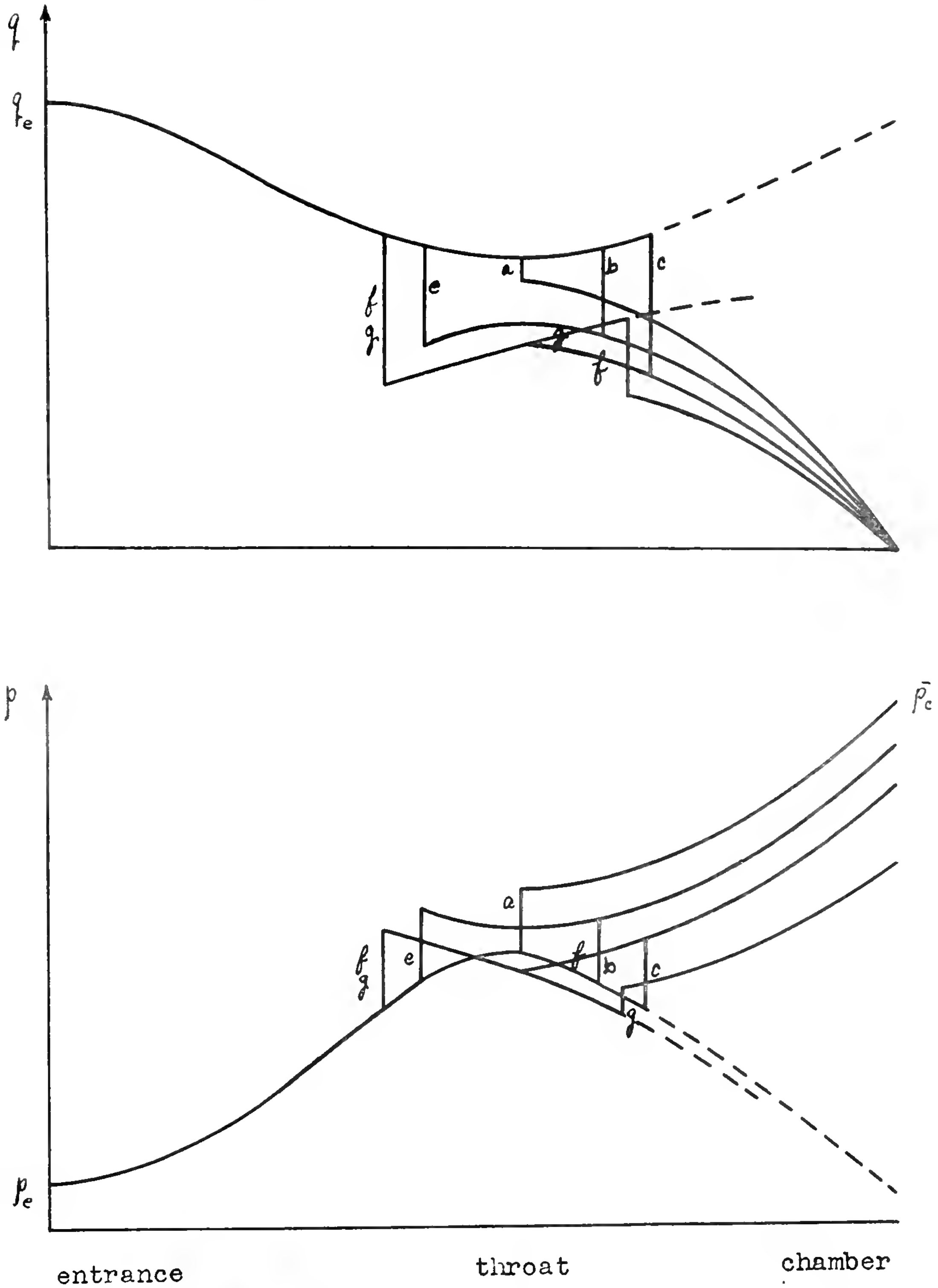
the flow rate will change. However, when the outside pressure is changed, the flow rate will not change, but the jet will adjust itself to the varied outside pressure by an appropriate pattern of shock fronts and rarefaction waves.

For the isentropic compressive flow, the response to changes in the entrance or chamber conditions is quite different. The compressor is designed to be operated with a certain ratio, p_e/p_c, of entrance pressure p_e to chamber pressure p_c and a certain Mach number, M_e, of the entrance flow. These particular values shall be referred to as the "ideal" ones. Naturally, we keep pressure and density of the entering air fixed. The quantities to be varied are thus the entrance speed q_e and chamber pressure p_c. Apparently, one obtains the clearest picture by first assuming that the entrance speed q_e is changed so as to be greater than the "ideal" entrance speed. It would be rather hopeless to appraise the influence of this change on the flow if one were not willing to consult the one-dimensional (hydraulic) flow theory and to assume accordingly that all shock fronts occurring in the compressor are plane and cover a whole cross-section. Experience with exha[redacted]hows that the results derived from such

[redacted]

[redacted] [redacted]

Figure 5



Distributi[redacted] speed and pressure in a compressor op[redacted] with an entrance speed greater than the ideal one and with various chamber pressures.

[redacted]

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assumptions represent quite well the cross-section average of what occurs in reality.[*] The hydraulic procedure[**] then indicates that the flow may pass the throat in a supersonic state and hence be a supersonic expansive flow past the throat. (Cf. Fig. 5 for this and the following discussion). In order that the flow speed reduces to zero in the chamber, a shock front may occur somewhere between throat and chamber (b, c). Where this shock occurs depends on the chamber pressure. When the chamber pressure is increased the shock front recedes toward the throat; the reason is that the shock then becomes weaker since the entropy jump decreases with increasing stagnation pressure (Cf. footnote on p. 46). The chamber pressure for

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* The actual pattern of shocks is more complicated and in general involves jet detachment. That the simplifying assumption is satisfactory is probably due to the re-attachment of the jet - a fact which has frequently been observed. Whether or not such re-attachment will occur in compressive flow is not certain.

** As seen from (1.08), q_e/q_* increases monotonically with M_e and as seen from (2.05), $h^2(q_e/q_*)$ increases monotonically with q_e/q_*; hence the ratio of critical to entrance cross-section area, $A_*/A_e = h^{-2}(q_e/q_*)$ decreases when $q_e = M_e c_e$ increases. When q_e assumes the ideal value the throat area A_t equals A_*, hence $A_t > A_*$ when q_e exceeds the ideal value. In that case, therefore, the flow is still supersonic when it reaches the throat.

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which the shock just stands in the throat (a) may be denoted by $\bar{p}_c = \bar{p}_c\ (q_e)$. What happens when the chamber pressure is increased beyond $\bar{p}_c$ seems difficult to conjecture. Probably pulsations in the flow will result. In any case the flow will be unstable when $p_c = \bar{p}_c$.

It should be observed that there is another possibility for the flow when $p_c < \bar{p}_c$, viz., a flow which passes through a shock in front of the nozzle, (i.e., between entrance and throat (e)), the flow then being subsonic when passing through the throat. One could imagine that such a forward position of a shock could be produced from a flow with a shock placed backward, by sending out, from the chamber against the stream, a short pressure pulse which is strong enough to push the backward shock over into forward position. If such a forward shock occurs and the chamber pressure is decreased, the shock, becoming stronger, will move away from the throat; for a certain chamber pressure, it will reach a limit position (f) which is defined by the condition that the flow speed is just sonic at the throat.(*) When the chamber pressure is further decreased,

* This limit position advances toward the entrance when the entrance speed is increased. It is interesting to note that for a compressor designed for the Mach number $M_e = 1.825$ this limit position reaches the entrance only when M_e has been increased to the value 4.4, as is easily calculated.

the shock remains in the limit position and the flow becomes supersonic past the throat (g). The section of the compressor beyond the shock front then acts as a forward nozzle and the adjustment to the chamber pressure will be achieved by a second shock. It is then clear that to every chamber pressure below $\bar{p}_c$ there are just two flows, one being supersonic at the throat and having a shock in backward position, the other being supersonic in front of the throat with a shock in forward position. The flow with a shock in forward position is probably the more stable one. At least it is hard to conceive how a shock in forward position could be pulled back through the throat by a signal sent out from the chamber.

Suppose now that the entrance velocity is again lowered to its ideal value. Then any shock in forward position or at the throat will vanish and the flow will become ideal. Or, if the chamber pressure p_c is less than the ideal one, (also denoted by $\bar{p}_c$), the flow will continue supersonically past the throat and adjust itself to a chamber pressure less than the ideal one by a shock in backward position.

When, further, the entrance speed q_e is decreased

below the ideal value, no steady flow can result, whatever the chamber pressure may be. The reason is that the critical cross-sectional area A_*, as determined from the hydraulic theory is then greater than the throat cross-sectional area and consequently critical sound speed has already been reached before the throat; (Cf. footnote on p. 48). Insertion of a shock in forward position would not help since, due to the increase of entropy across the shock, the critical cross-section would further increase. Probably the flow will become oscillatory, and if the chamber pressure is high enough, the stream might even reverse its direction and flow out of the chamber through the entrance opening.

In any case, it is clear that the <u>ideal flow is unstable</u>, since increase of the chamber pressure above $\bar{p}_c$ or decrease of the entrance speed will cause pulsations or another complete change of the flow.

Therefore, stable operation of the compressor requires an entrance speed q_e greater than the ideal speed for which the compressor was designed, and a chamber pressure p_c less than the maximum value $\bar{p}_c(q_e)$ which corresponds to the entrance speed q_e. Thus it is necessary to <u>admit a shock</u> in the operation of the compressor, but the shock strength may be kept small and its position can be confined

to the neighborhood of the throat.

10. <u>The Flow at the Compressor Entrance.</u>

Questions of particular interest arise when one considers projectiles, or projectile-shaped bodies, which carry a compressor of the type discussed in Section 8. We assume that the compressor is bored out of the nose of the projectile, as indicated in Figure 6. When such a projectile is opposed to a supersonic stream of air, the air has the opportunity of entering the compressor and

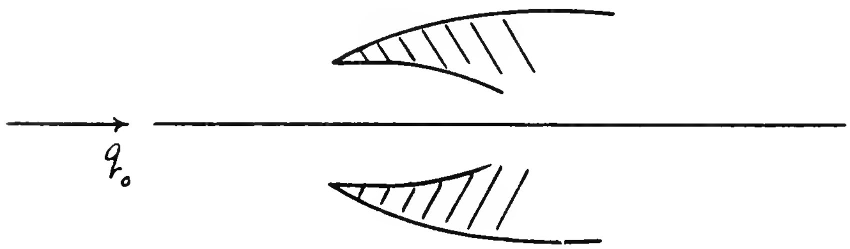

Fig. 6.

of arriving in the chamber isentropically compressed. Whether or not this phenomenon will really take place is the question to be discussed in the present section. (In Section 11, we shall discuss the question of the drag

resulting from the outer surface of the projectile).

Naturally, we assume that the compressor is a perfect one, in the sense of Section 8, since we know that otherwise a constant supersonic stream could not flow without interference of shocks. But even if the compressor is perfect and designed for the Mach number of the opposed air stream, it is not obvious that the air will enter the bore with constant axial velocity since the rim may cause a disturbance of the flow.

As is well known,(*) when a projectile with a conical nose moves with supersonic speed, a shock front will develop, which is either of conical shape and begins at the tip of the nose, or is curved and stands ahead of the projectile. Which of the two cases arises depends on the nose angle and on the Mach number M_o of the air flow relative to the projectile.

A conical shock will develop if the nose half-angle θ is small enough, i.e., is less than a certain extreme angle θ^c_{ext}, which depends on the Mach number M_o. For $M_o = 1.91$, e.g., the extreme cone angle is $\theta^c_{ext} = 39.5^o$, when $\gamma = 1.4$; when $M_o \to 1$, $\theta^c_{ext} \to 0^o$, and when $M_o \to \infty$, $\theta^c_{ext} \to 58^o$. These values can be read off from Busemann's

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* Cf. [13], [14], [15] and other literature quoted there.

([15] Fig. 7 and 8) or Maccoll's diagrams (Cf.[14],Fig.11). From these diagrams one can also read the shock angle β that corresponds to any nose-angle $\theta_o < \theta^c_{ext}$. For $M_o = 1.91$ and $\theta_o = 21^o$, e.g., one finds $\beta_o = 40^o$.

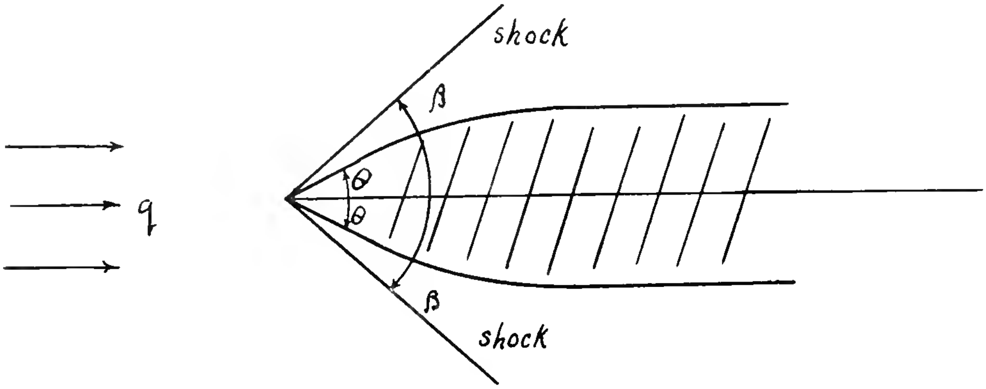


Fig. 7.

In case the nose angle is greater than the "extreme" angle, a curved shock front will stand ahead of the projectile. The position and the shape of this shock front depends not only on the Mach number and the nose angle, but also on where the conical part of the projectile tapers off; and quite generally on the shape of the projectile.

When the projectile carries a compressor of the type considered here, the exterior flow will not change very much except at the rim. The flow near the rim is

locally the same as the two-dimensional flow deflected by a wedge. Accordingly, the condition that the shock front begins at that rim and does not stand ahead of it, is the same as for the two-dimensional flow with reference to a wedge, viz., that the wedge angle θ is less than a certain extreme wedge angle θ^{w}_{ext}. This extreme angle

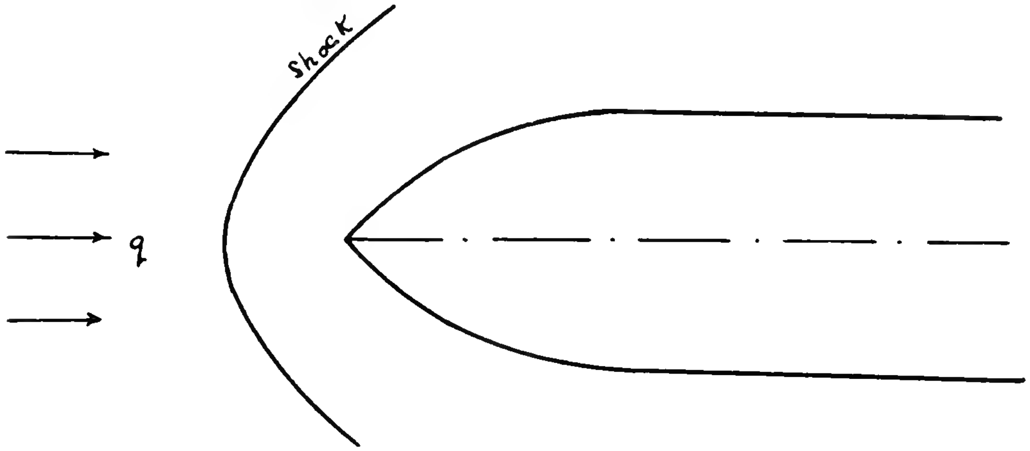


Fig. 8.

can also be read off from Busemann's diagrams. It decreases to zero when M_0 decreases to 1; it increases to 44.5° when M_0 increases indefinitely; its value for

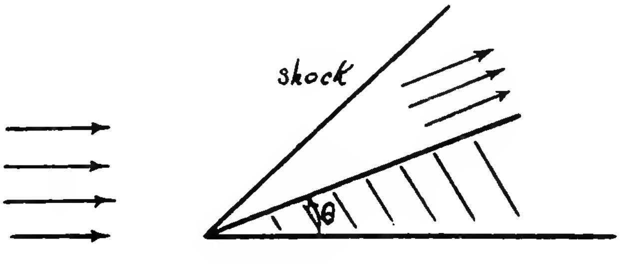


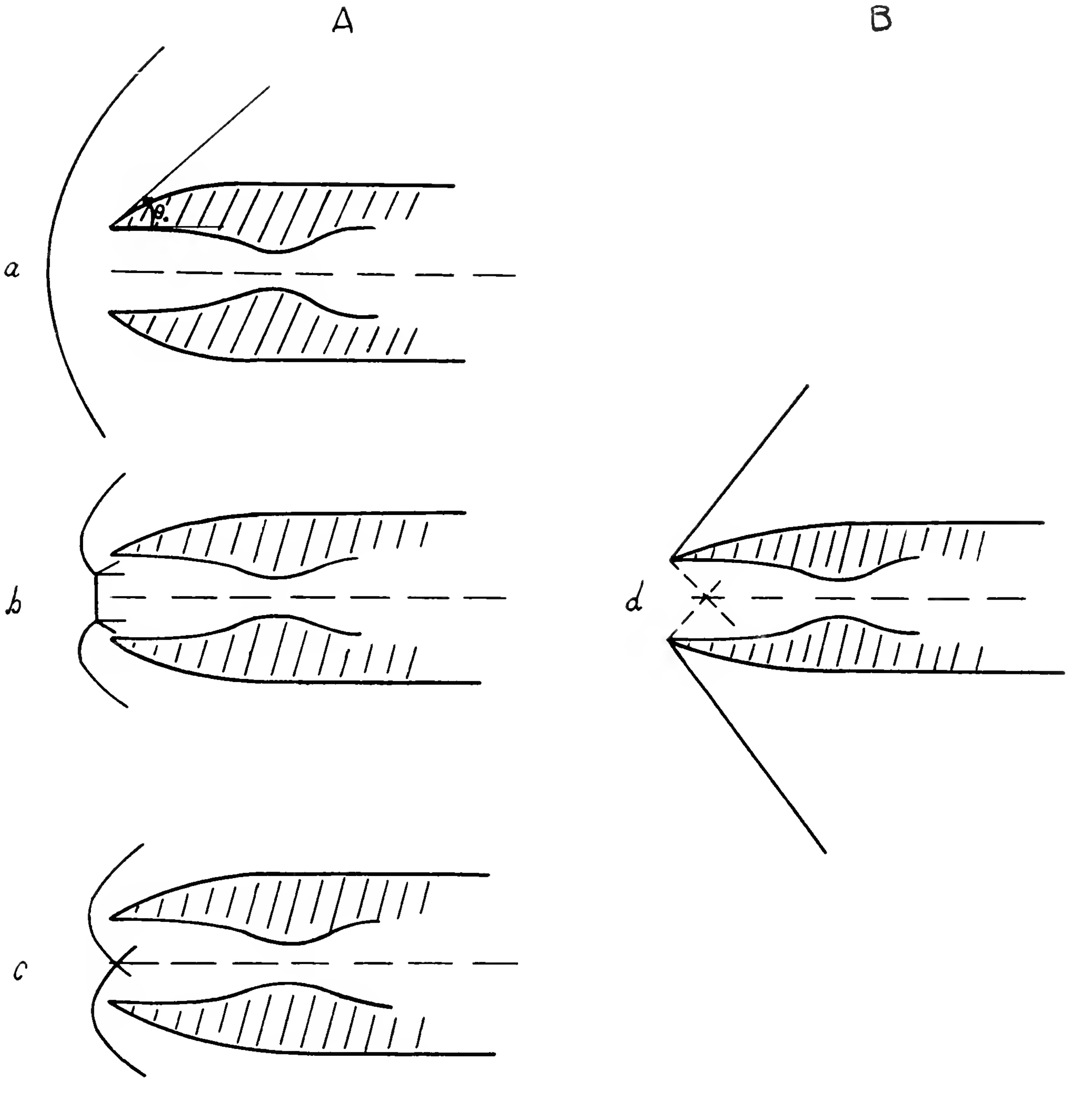
Fig. 9.

$M_0 = 1.91$, e.g., is $\theta_0 = 21^\circ$. Generally, it is less than the extreme angle for a cone; in rough approximation $\theta^w_{ext} = \frac{1}{2}\,\theta^c_{ext}$.

If the rim angle θ_0 is greater than the extreme angle, the shock front will stand either ahead of the projectile (Aa) (Cf. Figure 10) or ahead of the rim and be drawn into the projectile. There, either a disk-shaped "Mach-shock" perpendicular to the axis, followed by a complicated flow pattern, results (Ab) or the shock front forms a double cone (Ac). Which of these possibilities occurs depends on the size of the diameter of the compressor opening as compared with the distance the shock front stands ahead of the rim. If the compressor opening is relatively large, the drawn-in shock front may fade out and be practically sonic even before reaching the vertex of the double cone.

An entrance shock will, however, be completely avoided if the rim angle is less than the extreme angle, (i.e., less than 21° for $M_0 = 1.91$), provided the entrance to the compressor begins like a cylinder in axial direction, (as is the case for a perfect compressor); then only a sonic front will develop from the rim toward the interior. Thus the condition is found: <u>the supersonic stream will enter the perfect compressor with constant velocity</u>

Figure 10



$\theta_o > \theta_{ext.}$ $\theta_o < \theta_{ext.}$

Various shock configurations at the compressor entrance.

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without shock interference if the rim angle is less than the extreme angle for a wedge.

11. Remarks on the Drag of a Projectile Carrying a Compressor.

Suppose a projectile with a conical nose is opposed to a supersonic stream of air so strong that a shock front begins at the tip of the nose. Then the pressure on the nose, (which is constant over the surface), and hence the "pressure drag", i.e., the total force resulting from the pressure on the nose, can well be calculated on the basis of Maccoll's or Busemann's diagrams. As Maccoll has shown, such theoretical results agree surprisingly well with experimental results.

If, however, a compressor is bored out of the projectile, the tip of the nose is replaced by a rim and the flow along the outer surface of the projectile is no longer easily determined. Nevertheless, some rough estimates for the pressure drag can be derived.

It was explained in the previous section, that the shock front begins at the rim if the rim angle θ is the extreme angle for a wedge, θ^{w}_{ext}, (which is roughly one-half of the extreme angle for a cone, θ^{c}_{ext}). Suppose that this condition is satisfied; then a shock front S which behaves like that formed at a wedge begins at the

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rim. The angle β^w of this shock against the axis is greater than the angle β^c of a shock front, S^c, that would begin at the tip of the nose of an unmutilated cone with the same cone angle θ. (This fact can easily be read off from Busemann's diagram). Consequently, the shock S^w at the rim is stronger than the shock S^c. On the other hand one should expect that the shock front

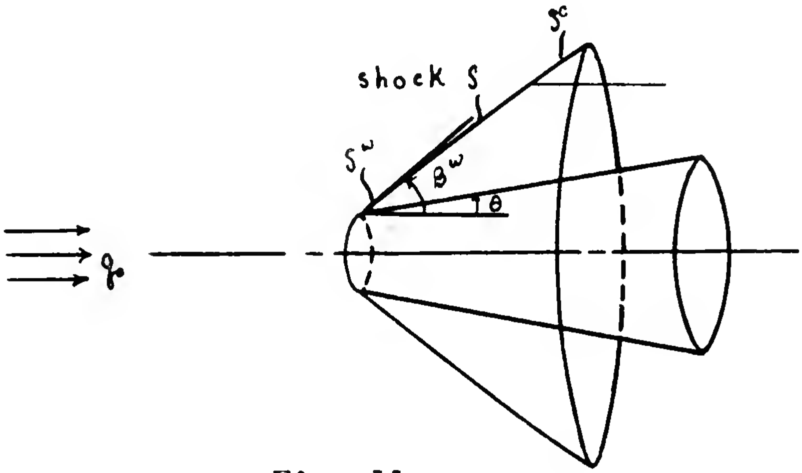

Fig. 11.

Flow against a cone with a bore.

S will gradually turn over into the direction of the shock front S^c, eventually running parallel to it.

At first sight one might also expect that, along the lateral surface of the cone, the flow will approach the flow F^c that would result from the unmutilated cone. This assumption, however, would not be correct. Since the shock $S = S^w$ at the rim is stronger than the

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shock S^c, the entropy increase across it is greater than the entropy increase across the shock S^c; and both entropy increases are completely independent of the cross-sectional area of the cylindrical bore. Consequently, as can be shown, the sound velocity at the lateral surface of the cone is larger while the air velocity there is smaller than the corresponding values for the flow F^c, independent of the size of the bore.

If the conical section of the projectile is sufficiently long, these quantities will differ from those for the flow F^c in the layer whose width approaches zero. As in the case of a boundary layer induced by viscosity, one can argue that the pressure will be constant across such a layer. Hence, when the size of the hole shrinks to zero, the pressure will approach that for the flow F^c while the values of density and air velocity will remain smaller by a finite amount than the corresponding values for flow F^c. If the pressure did not behave in the indicated way, the "pressure drag" would change abruptly when the hole disappeared; this would be absurd.

Since the shock $S = S^w$ at the rim is stronger than the shock S^c the pressure p^w immediately

behind it is greater than p^c behind the shock front S^c. The pressure along the side surface of the bored cone will probably decrease monotonically from $p = p^w$ to $p = p^c$. Generally we have there

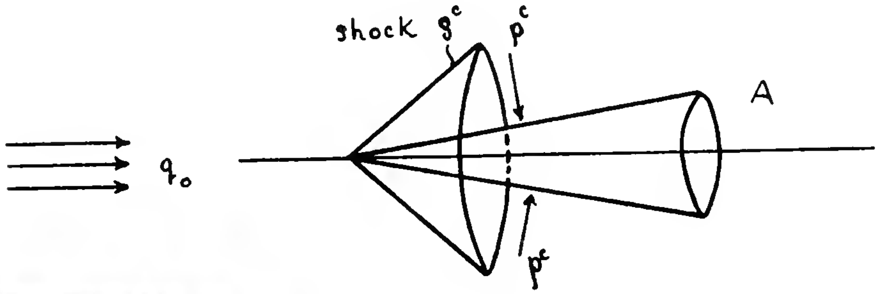
$$p^w > p > p^c.$$

The values of p^w and p^c can easily be determined from Busemann's diagrams.

The preceding remarks lead to rough estimates for the pressure drag against a section of the cone with a bore. For a complete cone the pressure, p^c, is constant and the drag is simply

$$D^c = p^c A ,$$

where A is the area of the base of the cone, when the cone is cut perpendicular to the axis at the end of the section considered.



[redacted] Fig. 12.

Flow against a cone.

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Consider now the cone with a bore of entrance cross-sectional area a.

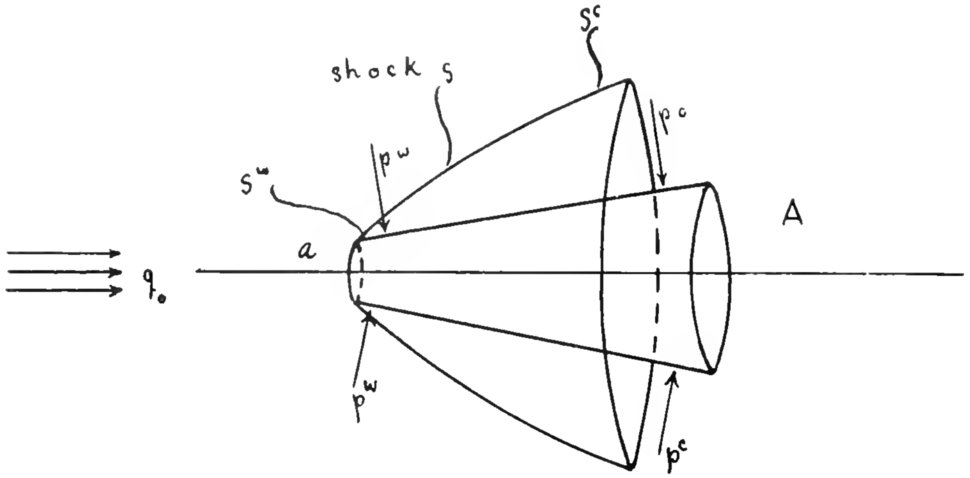

Fig. 13.
Flow against a cone with a bore

From the inequality $p^w > p > p^c$ we derive for the pressure drag

$$p^c\,(A-a) < D < p^w\,(A-a).$$

If the cone is relatively long, i.e., if $a < A$, the value of D is probably nearer to the lower limit. If, however, the cross-section of the bore is relatively large, or if the obstacle deviates soon from the conical shape and tapers off to approach a cylinder, then the

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upper limit $p^w(A-a)$ is probably a better approximation to the correct value of D.

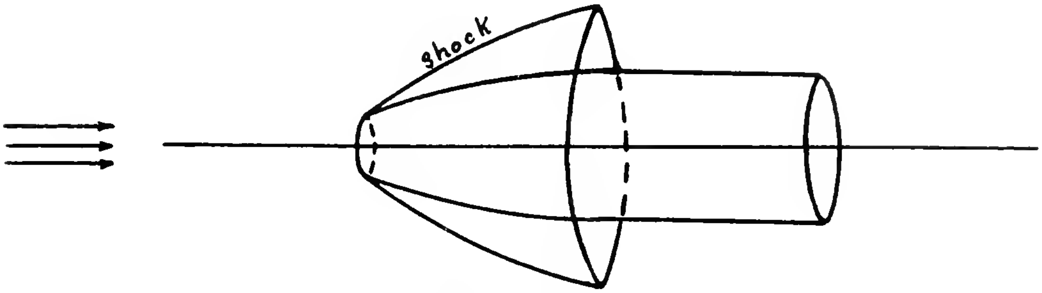

Fig. 14.

As an example, let us assume that the incoming flow has the Mach number 1.91 and that the cone angle θ_o is 9.1^o. Then we find from Busemann's diagrams that

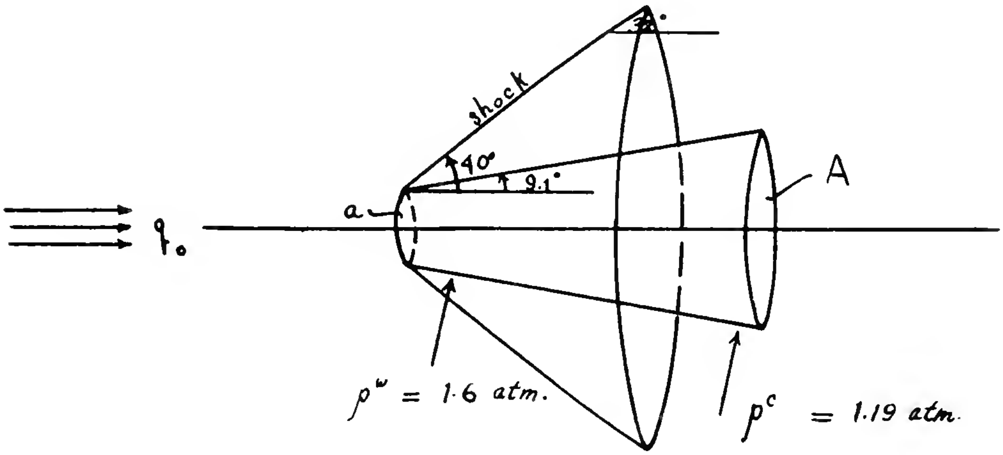

Fig. 15.
Flow against a cone with a bore

the shock angle for a wedge with $\theta_o = 9.1^o$ is $\beta^w = 40^o$, while the shock angle for a cone with $\theta_o = 9.1^o$ is $\beta^c = 32^o$. Hence the shock front will begin at the rim

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with an angle of 40° and eventually turn over so as to make an angle of 32° with the axis. Further one derives from Busemann's diagrams

$$p^{w}/p_{o} = 1.6 \quad \text{and} \quad p^{c}/p_{o} = 1.19 \ ,$$

p_o being the pressure of the air in front of the shock front.

Hence the pressure drag should lie between

$$1.2\ p_{o}(A-a) \quad \text{and} \quad 1.6\ p_{o}(A-a) \quad .$$

The wide margin of uncertainty in these appraisals could be narrowed down only by a more refined treatment.

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APPENDIX

12. Mathematical Details to Part I Concerning Isentropic Flow Through Nozzles.

In this section we shall supply various calculations omitted in the main sections of Part I.

First we shall give a detailed derivation of the approximate formulas (2.11) and (2.15) from the differential equations (2.01) and (2.02).

When ξ and η, instead of ϕ and ψ [Cf.(2.04) and (2.05)], are introduced as new variables these equations lead to

$$(12.01)\quad \begin{cases} \dfrac{\partial x}{\partial \xi} = (q_o/q)\cos\theta\,, & \dfrac{\partial x}{\partial \eta} = -(\eta/y)\,h^2 \sin\theta \\[2ex] \dfrac{\partial y}{\partial \xi} = (q_o/q)\sin\theta\,, & \dfrac{\partial y}{\partial \eta} = (\eta/y)\,h^2\cos\theta, \end{cases}$$

as is immediately verified. The initial conditions for the new dependent variables are

$$(12.02)\qquad x = \xi\ ,\ y = 0 \quad \text{on the axis } \eta = 0.$$

Elimination of x and y from equations (12.01) yields

the two equations

$$(12.03)\quad \begin{cases} \dfrac{\partial(q_o/q)}{\partial\eta} = -(\eta/y)h^2\,\dfrac{\partial\theta}{\partial\xi}\,, \\[2ex] \dfrac{\partial\theta}{\partial\eta} = (q/q_o)\,\dfrac{\partial(\eta/y)h^2}{\partial\xi} \end{cases}$$

which together with

$$(12.03)_o\qquad \frac{\partial y}{\partial\eta} = (\eta/y)h^2\cos\theta$$

serve to determine q, θ, y from the initial values

$$(12.04)\quad q = q_o(\xi),\ \theta = 0,\ y = 0 \quad \text{on the axis } \eta = 0.$$

The expansions of these quantities with respect to powers of η have the form

$$(12.05)\quad \begin{cases} y = y_1\eta + y_3\eta^3 + \dots \\[1ex] \theta = \theta_1\eta + \dots \\[1ex] q/q_o = 1 + \delta_2\eta^2 + \dots \end{cases}$$

From the last expansion one derives, by (2.06) and (2.07)

$$h = h_o \left\{ 1 + \tfrac{1}{2} [M_o^2 - 1] \delta_2 \eta^2 + \ldots \right\}$$

and by (1.05) and (1.08),

$$p = p_o \left\{ 1 - \gamma M_o^2 \delta_2 \eta^2 + \ldots \right\} .$$

Insertion in (12.03) yields

$$(12.06) \begin{cases} y_1 = h_o(\xi) \\ \theta_1 = h_o'(\xi) \\ \delta_2 = \tfrac{1}{2} h_o(\xi) h_o''(\xi) \\ y_3 = \tfrac{1}{8} [M_o^2(\xi) - 1] h_o(\xi) h_o''(\xi) - \tfrac{1}{8} [h_o'(\xi)]^2 . \end{cases}$$

For the expansion

$$x = \xi + x_2 \eta^2 + \ldots$$

x

one then finds from $\partial \psi / \partial \eta = -(\eta / y) h^2 \sin \theta$,

$\frac{\partial x}{\partial \eta}$

$$x_2 = -\tfrac{1}{2} h_o(\xi)\, h_o'(\xi) .$$

Insertion in (12.05) gives the expansions (2.11) to (2.15).

Secondly, we shall supply the steps necessary for

the calculations of the <u>thrust</u>, Section 5. We introduce the variable η in the integral (5.02), which is to be extended over a line ξ = const., and employ the following relations:

$$\psi d\psi = \rho_* q_* \eta \, d\eta$$

from (2.04), and

$$y dy = h^2 \cos^2 \theta \; \eta d\eta$$

$$= \rho_* q_* \rho^{-1} q^{-1} \cos \theta \; \eta d\eta$$

from $(12.03)_0$ and (2.06).

Furthermore, we use

$$(12.07) \qquad \rho_*^{-1} p_* = \frac{1}{\gamma} q_*^2 = \frac{1-\nu}{1+\nu} q_*^2$$

$$(12.08) \qquad \rho^{-1} p = \frac{1}{1+\nu} [q_*^2 - \nu q^2]$$

from (1.02), (1.04) and $c_* = q_*$

and

$$A_* = \pi \eta^2 \quad .$$

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Thus

$$K_1 = \frac{1+\nu}{1-\nu}\,\frac{2}{\eta^2}\int_0^{\eta}(q/q_*)\cos\theta\;\eta d\eta$$

$$+\frac{1}{1-\nu}\,\frac{2}{\eta^2}\int_0^{\eta}[\,q_*/q - \nu q/q_*\,]\cos\theta\;\eta d\eta$$

$$= \frac{1}{1-\nu}\,\frac{2}{\eta^2}\int_0^{\eta}[\,q/q_* + q_*/q\,]\cos\theta\;\eta d\eta\ .$$

Using the expansions (2.13) and (2.14) up to second order in η , one obtains

$$K_1 = \frac{1}{1-\nu}\,[(q_o/q_* + q_*/q_o)(1 - \tfrac{1}{4}[h_o']^2\eta^2)$$

$$+\ \tfrac{1}{4}(q_o/q_* - q_*/q_o)\ h_o h_o''\,\eta^2\ ]\ .$$

It is easily verified that the expression given in (5.11) possesses the same expansion up to second order in η^2 (or up to third order, since no third order terms occur).

The expansion of the expression K_a given by (5.08), is found from (2.12),

$$K_a = (p_a/p_*)h_o^2\left\{1 + \tfrac{1}{4}\,[(M_o^2 - 1)\ h_o h_o'' - (h_o')^2\,]\,\eta^2\right\}\ .$$

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It agrees, up to second (or third) order, with the expansion of the expression given in (5.12). Thus the basic approximate expression (5.13) for the thrust is established.

A proof may be given that the <u>exact thrust</u> F as a function of the cut-off point of a given contour is a maximum when the pressure at the cut-off point coincides with the outside pressure. To this end we make use of identities

$$\frac{\partial y^2}{\partial \xi} = 2(q_o/q)\ y \sin \theta$$

and

$$\frac{\partial}{\partial \xi}\ \frac{1}{1-\nu}\ (q/q_* + q_*/q) \cos \theta\, \eta = 2 \frac{\partial}{\partial \eta}(p/p_*)(q/q_*)y \sin \theta,$$

which can be verified by using relations (12.01), (12.02), (12.07) and (12.08). Integration with respect to η yields, using (5.07) and (5.08)

$$\frac{\partial}{\partial \xi}\ K_1 = \frac{2\, p\, q_o y \sin \theta}{p_* q}$$

$$\frac{\partial}{\partial \xi}\ K_a = \frac{2\, p_a q_o\ y \sin \theta}{p_* q}$$

where it is understood that the integral is evaluated at the nozzle contour.

Hence

$$\frac{\partial}{\partial \xi} K = 2\left(\frac{p - p_a}{p_*}\right)\left(\frac{q_o}{q}\right) \; y \sin \theta .$$

As a result it follows that K is stationary when at the cut-off point either $p=p_a$ or $\theta = 0$; the latter condition would necessitate cutting the contour at the throat. That K assumes a maximum in the former case and a minimum at the throat could easily be shown. We refer to a different presentation in an earlier memorandum [8].

13. Construction of a Perfect Three-Dimensional Flow from any Expanding Flow with Axial Symmetry.

It was shown in Section 7 how any expanding two-dimensional flow with axial symmetry can be turned into a constant parallel flow. In this section we shall explain how the same problem can be solved for three-dimensional flow.

Suppose the flow is to emerge with the speed q_o. Then one determines the point A_o on the axis at which this speed is reached. Through this point, the Mach cone

C_o is drawn in upstream direction. The original flow is then to be retained only up to this cone C_o. Further one draws through this point A_o in downstream direction, the straight cone D_o whose angle against the axis is the Mach angle belonging to the speed q_o. Along this cone, all quantities characterizing the flow are constant; their values are therefore the same as at the point A_o. Since these quantities are given along both cones C_o and D_o, the flow in between can be determined by solving the partial differential equations of the flow problem. Mathematically speaking, the problem is an "initial value problem" for a hyperbolic differential equation, the initial values being given on two "characteristic lines."(*)

To this end the basic differential equations (2.01) and (2.02) will be re-written by referring them to characteristic parameters,(**) i.e., to variables which are constant along the Mach lines. Instead of the speed

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* Cf. these notions [2] Ch. 5 and [16].

** The method explained in the following can also be employed for a variety of other three-dimensional supersonic flow problems.

q we introduce the angles η, δ, and ζ by

(13.01) $$\tan\eta = \sqrt{M^2 - 1}$$

(13.02) $$\tan\delta = \mu\tan\eta$$

and

(13.03) $$\zeta = \mu^{-1}\delta - \eta \quad .$$

Differentiating (13.03) we obtain

$$d\zeta = \mu\cos^2\delta \; d\tan\delta - \cos^2\eta \, d\tan\eta$$

$$= (\cos^2\delta - \cos^2\eta) d\tan\eta$$

or since by (1.08)

$$d\tan^2\eta = \frac{1}{(1-\mu^2)\cos^2\delta\,\cos^2\eta}\;\frac{2dq}{q}$$

(13.04) $$d\zeta = \tan\eta\,\frac{dq}{q}$$

and by (1.07)

(13.05) $$\frac{d(\rho q)}{\rho q} = -\tan\eta\, d\zeta \quad .$$

It is clear that q may be considered a function

of ζ and consequently, also η, δ, c, p, and ρ may be considered functions of ζ. The latter three quantities are easily expressed in terms of δ. This is seen from the relation

$$\cos^2\delta = \frac{q_*^2 - \mu^2 q^2}{(1-\mu^2)q_*^2},$$

which by (1.04), (1.05) and (1.06) leads to

$$(13.06) \qquad \frac{c}{c_*} = \cos\delta \;, \; \frac{p}{p_*} = \cos^{2\nu+1}\delta \;, \; \frac{\rho}{\rho_*} = \cos^{2\nu}\delta \,.$$

Further, one easily verifies

$$(13.07) \qquad \frac{q}{q_*} = \frac{\sin\delta}{\mu\sin\eta} = \frac{\cos\delta}{\cos\eta} \,.$$

When we now eliminate ϕ or ψ from equations (2.01) and (2.02) by expressing the identities(*)

$\phi_{xy} - \phi_{yx} = 0$, $(\psi^2)_{xy} - (\psi^2)_{yx} = 0$ in terms of the right-hand members of these equations, we obtain a simple form of the result by virtue of (13.04) and (13.05)

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* Subscripts x and y indicate partial differentiation with respect to these quantities.

$$
(13.08)\begin{cases}
[\cot \eta\xi_y - \theta_x]\cos\theta - [\cot \eta\xi_x + \theta_y]\sin\theta = 0 \\
[\tan \eta\xi_y - \theta_x]\sin\theta + [\tan \eta\xi_x - \theta_y]\cos\theta - y^{-1}\sin\theta = 0.
\end{cases}
$$

To handle our problem it is natural to introduce characteristic parameters σ and τ since then the two boundary cones C_o and D_o are given by σ = const. and τ = const. respectively. We assume in particular that C_o is given by $\sigma = 0$ and D_o by $\tau = 0$.

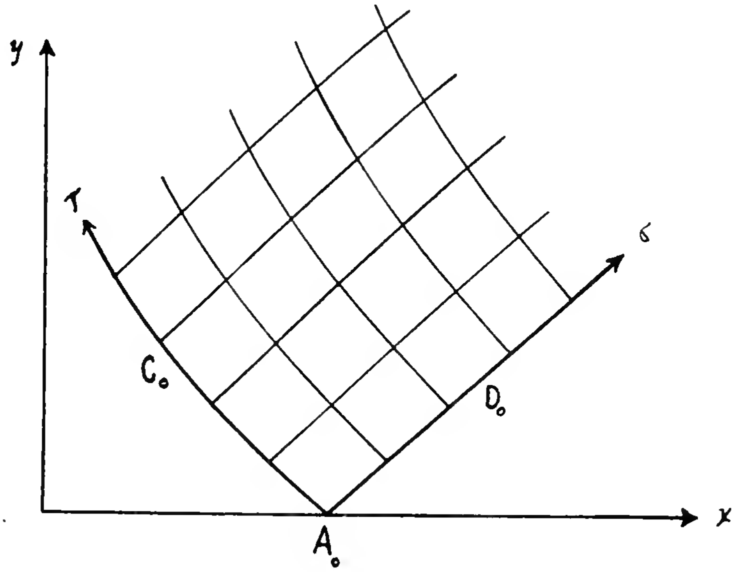


Fig. 16.

The characteristic directions are then characterized by the condition that they make the Mach angle α with the flow direction or the angle η with the normal to that direction. We count σ and τ positive away from the axis as indicated in Fig. 17.

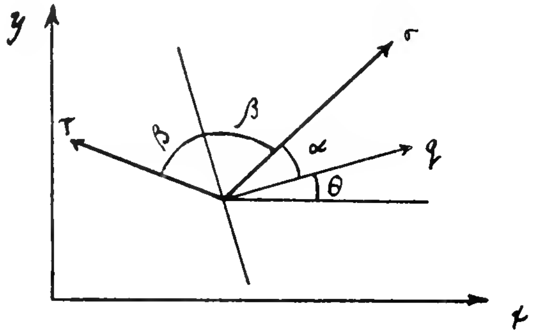


Fig. 17.

Instead of derivatives $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\tau}$ we employ differentials d_σ and d_τ for the corresponding directions. The equations for x and y then are

(13.09) $$d_\sigma x = S\, d_\sigma y, \quad d_\tau x = -T\, d_\tau y$$

with

(13.10) $$S = \tan(\eta - \theta), \quad T = \tan(\eta + \theta).$$

Equations (13.08) are, by virtue of (13.09), equivalent with

([redacted] $\theta) = P d_\sigma y, \quad d_\tau(\xi + \theta) = -Q\, d_\tau y$

[redacted]

where

$$
(13.12)\begin{cases}
P = \dfrac{1}{y}\,\dfrac{\cos\eta \sin\theta}{\cos(\eta-\theta)} = \dfrac{1}{y}\;\dfrac{1}{\cot\theta + \tan\eta} \\[2ex]
\quad = y^{-1}\sin\theta\,[\cos\theta - S\sin\theta] \\[2ex]
Q = \dfrac{1}{y}\,\dfrac{\cos\eta \sin\theta}{\cos(\eta+\theta)} = \dfrac{1}{y}\;\dfrac{1}{\cot\theta - \tan\eta} \\[2ex]
\quad = y^{-1}\sin\theta\,[\cos\theta + T\sin\theta] \quad .
\end{cases}
$$

Relations (13.11) can easily be verified; their form exhibits the fact that (13.09) determines characteristic directions.(*)

---

* From the formulas (13.01) to (13.12) one easily derives the upper limit for the entrance curvature as given in Sec.7. The relation which one obtains from (13.09) by elimination of x reduces to

(1) $\quad 2\tan\eta_0 d^2_{\sigma\tau}\,y + \sec^2\eta_0 d_\tau(\eta - \theta)d_\sigma y = 0$

on the line D_0. From $d\zeta = (\cos^2\delta - \cos^2\eta)\,d\tan\eta =$ $(1-\mu^2)\cos^2\delta\,\tan^2\eta\,d\eta$ or $d\eta = (1-\mu^2)^{-1}(\cot^2\eta + \mu^2)d\zeta$, and $d_\tau\zeta = -d_\tau\theta$ on D_0, one derives $d_\tau(\theta - \eta) = (1-\mu^2)^{-1}\csc^2\eta_0$ where the parameter τ is so chosen that $d_\tau\theta = 1$ on D_0. If the Mach lines D converge at the point A_0 we have $d_\tau y = 0$ there, and relation (1) can be integrated to give

(2) $2\tan\eta_0 d_\tau y = (1-\mu^2)^{-1}\csc^2\eta_0\sec^2\eta_0\,y = \Gamma_0\,y.$

On the other hand, along a streamline through a point of D_0 we have $dy = 0$, hence $ds = -dx = 2\tan\eta_0 d_\tau y$; further $d\theta = d_\tau\theta$. Therefore, the radius of curvature is $R = ds/d\theta =$ $2\tan\eta_0 d_\tau y = \Gamma_0 y$. If the Mach lines do not converge on D_0 we obviously have $R > \Gamma_0\,y$.

[redacted]

From relations (2.01) and (2.02) together with (13.06) and (13.07) one obtains

$$(13.13)\quad d_\sigma \phi = \frac{q_*}{\mu} \frac{\sin\delta}{\cos(\eta-\theta)} d_\sigma y; \quad d_\tau \phi = -\frac{q_*}{\mu} \frac{\sin\delta}{\cos(\eta+\theta)} d_\tau y$$

$$(13.14)\quad d_\sigma \psi^2 = \rho_* q_* \frac{\cos^{2\nu+1}\delta}{\cos(\eta-\theta)} d_\sigma y^2; \, d_\tau \psi^2 = \rho_* q_* \frac{\cos^{2\nu+1}\delta}{\cos(\eta+\theta)} d_\tau y^2.$$

As initial conditions we have

$$(13.15)\quad y = \cot \eta_o (x - x_o), \ \theta = 0, \ \zeta = \zeta_o, \ \phi = \frac{q_*}{\mu_*} \frac{\sin \delta_o}{\cos \eta_o} y$$

$$\psi^2 = \rho_* q_* \frac{\cos^{2\nu+1} \delta_o}{\cos \eta_o} y^2,$$

where η_o, δ_o, and ζ_o are determined from (13.01), (13.02), and (13.03) for $q = q_o$.

Along C_o we have the condition that ϕ, ψ, θ, q and hence η, δ, ζ are those functions of x and y that characterize the incoming stream. A further relation between x and y (η and θ) is given by the second relation (13.09).

We note that the differential equations (13.11) are rather singular at A_o. On D_o, i.e., $\tau = 0$, we have

[redacted]

$\theta = 0$ and hence $P = Q = 0$. On C_o, i.e., $\sigma = 0$, the ratio $\frac{\sin \theta}{y}$ approaches a value different from zero, hence P and Q do not approach zero as $\tau \to 0$ on $\sigma = 0$. Nevertheless it is likely that the problem has a unique solution.

To obtain the solution numerically we intend to employ the method of finite differences. We place a net of lines σ = const. and τ = const. across the sector between C_o and D_o. By Δ_σ and Δ_τ we then indicate the differences of values at neighboring net points in the σ and τ direction, respectively. More specifically let z be a quantity at one net point; then we denote by z^+ and z^- the values of that quantity at the next net points reached by going backwards in σ or τ direction respectively.

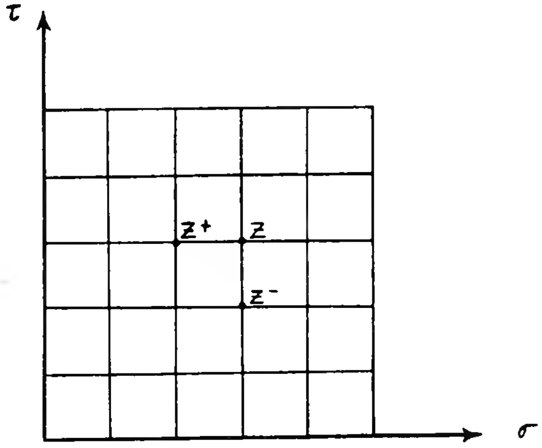


Fig. 18.

We then set $\Delta_\sigma z = z - z^+$ and $\Delta_\tau z = z - z^-$

$$(13.16)\qquad \begin{cases} S = \frac{1}{2}(S^+ + S),\ T = \frac{1}{2}(T^- + T) \\ \\ P = \frac{1}{2}(P^+ + P),\ Q = \frac{1}{2}(Q^- + Q) \end{cases}$$

Then we replace the differential equations by the difference equations:

$$(13.17)\qquad \Delta_\sigma x = \tilde{S}\,\Delta_\sigma y\ ,\qquad \Delta_\tau x = -\tilde{T}\,\Delta_\tau y$$

$$(13.18)\qquad \Delta_\sigma(\zeta - \theta) = \tilde{P}\,\Delta_\sigma y\ ;\ \Delta_\tau(\zeta + \theta) = -\tilde{Q}\,\Delta_\tau y$$

$$(13.19)\qquad \Delta_\sigma \phi = \frac{q_*}{\mu}\left(\frac{\widetilde{\sin\delta}}{\cos(\eta - \theta)}\right)\Delta_\sigma y;\quad \Delta_\tau \psi = -\frac{q_*}{\mu}\left(\frac{\widetilde{\sin\delta}}{\cos(\eta + \theta)}\right)\Delta_\tau y$$

$$(13.20)\qquad \Delta_\sigma \psi^2 = \rho_* q_*\left(\frac{\widetilde{\cos^{2\nu+1}\delta}}{\cos(\eta - \theta)}\right)\Delta_\sigma y^2;\quad \Delta_\tau \psi^2 = \rho_* q_*\left(\frac{\widetilde{\cos^{2\nu+1}\delta}}{\cos(\eta + \theta)}\right)\Delta_\tau y^2\ .$$

We eliminate x from equation (13.17) and obtain

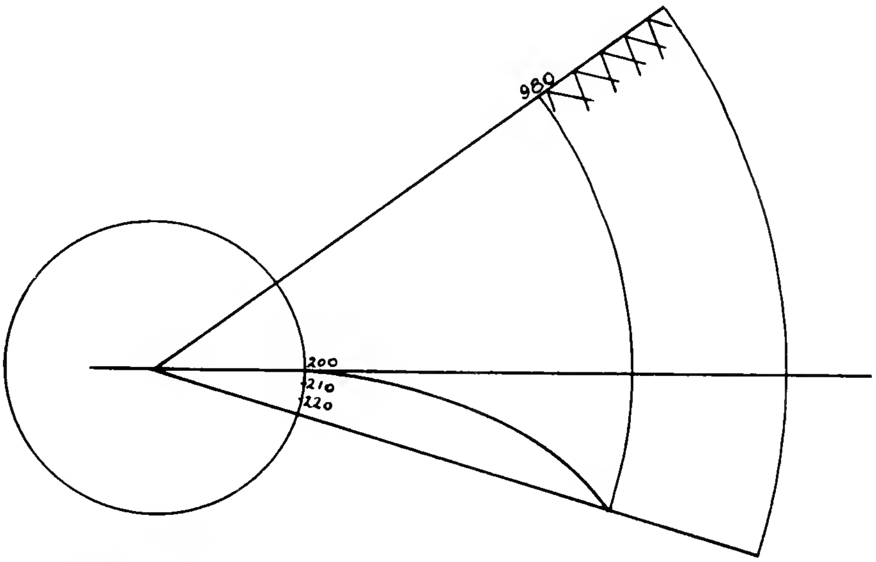
$$(13.21)\qquad y = (x^- - x^+ + \tilde{S}\,y^+ + \tilde{T}\,y^-)\Big/(\tilde{S} + \tilde{T})\ .$$

Further, from (13.18) we have

$$(13.22)\qquad 2\zeta = \zeta^+ + \zeta^- - (\theta^+ - \theta^-) + \tilde{P}\,\Delta_\sigma y - \tilde{Q}\,\Delta_\tau y$$

$$(13.23) \quad 2\,\theta = \theta^{+} + \theta^{-} - (\xi^{+} - \xi^{-}) - \tilde{P}\,\Delta_{\sigma} y - \tilde{Q}\,\Delta_{\tau} y$$

The procedure is then as follows: Suppose the values for (+) or (-) are known; then one first replaces $\tilde{S}$, $\tilde{T}$, $\tilde{P}$, $\tilde{Q}$, by S^{+}, T^{-}, P^{+}, Q^{-} respectively and determines y from (13.21), x from (13.17) or (13.18), ξ and θ from (13.22) and (13.23). From ξ one is to find η . To this end one may employ a diagram or a table by Busemann ([5], pp. 424 and 426) provided one is willing to set $\gamma = 1.405$. How to do this may be shown by an example. Suppose $\xi = 21^{\circ}$ is found. Then one follows the epicycloid in Busemann's diagram [5] until its polar angle is 21°. The corresponding point has a distance from the origin which belongs to the "pressure number" 979. In Busemann's



[redacted] Fig. 19.

[redacted] [redacted]

Table I one then reads off $q/c = w/q = 1.815$. This number is $\sec\eta$ hence η can be determined. Having found η one can determine S, T, P, Q. Now one is able to calculate the average values; and, employing them, one can repeat the process until the results do not change appreciably. The scheme corresponds to Moulton's method for solving differential equations. It seems necessary to follow this complicated scheme in the present case, at least near the point A_o, in view of the strong singularity there.

Since all desired quantities are given on C_o and D_o it is clear that the solution can be determined in all net points within the sector between C_o and D_o.

<u>Example</u>

As an example, we consider the case that the incoming flow is purely radial. Such a flow is characterized by the solutions

$$(13.24) \qquad x = r\cos\theta,\ y = r\sin\theta;$$

where $r = r(q)$ is given by

$$(13.25) \qquad r = r_* \sqrt{q_* \rho_* / q\rho} = r_* h;$$

r_* is the distance from the origin at which the critical

[redacted]

[redacted]

speed is reached; q/q_* , ρ/ρ_* are given by (13.06) and (13.07) . One easily verifies that relations (13.08) are then satisfied, if one observes that $r^{-1}dr = \frac{1}{2}\tan\eta \, d\xi$ holds by virtue of (13.25) and (13.05). Also we find

$$\psi^2 = 2r_*^2 \rho_* q_* (1 - \cos\theta) \tag{13.26}$$

and

$$\phi = \int_{r_*}^{r} q(r)dr . \tag{13.27}$$

To obtain the equation of the characteristic D_o we insert (13.24) in the second equation (13.09) and find $r^{-1}dr = -\tan\eta \, d\theta$. Comparison with $r^{-1}dr = \frac{1}{2}\tan\eta \, d\xi$ yields

$$\theta = \frac{1}{2}(\xi_o - \xi), \tag{13.28}$$

ξ_o being the value of ξ at the point at which the characteristic intersects the axis. (It is interesting to note that the equation of the characteristic for two-dimensional flow would be $\theta = \xi_o - \xi$; for three-dimensional flow, therefore, the same state as for two-dimensional flow is reached on a characteristic on turning through half the angle.) To a given value of θ one

determines ζ , then reads off from Busemann's diagram [5] the "pressure number" and takes the values of $q_* \rho_* / q \rho$ which give r/r_* by (13.25). All other quantities are then readily obtained.

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In these papers, special flows through nozzles are determined. Meyer finds two-dimensional flows by a development with respect to powers of the coordinates x and y, assuming that the velocity varies linearly in the axial direction. Taylor obtains symmetrical and anti-symmetrical flows by similar power series expansions, satisfying boundary conditions at a given circular nozzle contour. He finds that the symmetrical,flow ceases to exist when the velocity in the throat reaches a certain magnitude. Taylor's investigations have been refined by Görtler. Braun uses a Rayleigh-Ritz method based on a variational problem formulated by Bateman. Although the expression to be varied has no extremum the formalism of the Rayleigh-Ritz method can be used. Braun's method is carried through for the flow past a cylinder; it could be used also for nozzle flow.

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